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A Study of the Three-Dimensional Quasi-Geostrophic System

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A Study of the Three-Dimensional Quasi-Geostrophic System

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A Study of the Three-Dimensional Quasi-Geostrophic System

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The three-dimensional quasi-geostrophic system is a set of equations used to model atmospheric and oceanic circulation. We shall consider both inviscid and viscous variations of the underlying model, which may be posed on a variety of spatial domains. As is typical of models from fluid mechanics, questions of well-posedness are physically relevant and mathematically interesting. In this work, we study physical properties of the quasi-geostrophic system using modern tools from pure and applied mathematics including regularity theory, asymptotic analysis, and convex integration.

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Chapter 1

Introduction to Quasi-Geostrophic Flow

1.1 The Three-Dimensional Quasi-Geostrophic System

Atmospheric and oceanic circulation is described by the incompressible Navier-Stokes equations. For the time being, we pose these equations in the upper half space $\mathbb{R}^3_+$, where the boundary at $z = 0$ represents the boundary between the atmosphere and the surface of the Earth. The unknowns include the fluid velocity $\vec{u} = (u, v, w)$, which we write as

$$\underset{(t,x,y,z)\to\vec{u}(t,x,y,z)}{\vec{u} : [0, \infty) \times \mathbb{R}^3_+ \to \mathbb{R},}$$

the pressure $p : [0, \infty) \times \mathbb{R}^3_+ \to \mathbb{R}$, and the density $\rho : [0, \infty) \times \mathbb{R}^3_+ \to \mathbb{R}$. The equations then take the form [42]

$$(NSE) \qquad \begin{cases} \partial_t u + \vec{u} \cdot \nabla u - f_0 v + (\rho_s)^{-1} \left(\partial_x p + \mathcal{F}(u)\right) = 0 \\ \partial_t v + \vec{u} \cdot \nabla v + f_0 u + (\rho_s)^{-1} \left(\partial_y p + \mathcal{F}(v)\right) = 0 \\ \partial_t w + \vec{u} \cdot \nabla w + (\rho_s)^{-1} \left(\partial_z p + \mathcal{F}(w) + g\rho\right) = 0 \\ \partial_t \rho + \vec{u} \cdot \nabla \rho = 0 \\ \nabla \cdot \vec{u} = 0. \end{cases}$$

Here g is the acceleration due to gravity and f_0 encodes the speed of the rotation of the Earth. The terms $\mathcal{F}(\vec{u})$ represent turbulent viscosities and may be either zero, in which case the equations are known as the incompressible Euler equations, or non-zero terms that we shall describe later. We have employed the Boussinesq approximation above by ignoring density variations except when amplified by the force of gravity (hence the term $g\rho$ appears only in the equation for w). In addition, we have included the Coriolis forcing terms $-f_0 v$ and $f_0 u$ in the equations for u and v. The first three equations are *convective* equations representing the conservation of momentum. The fourth is the *continuity* equation, representing conservation of mass. The final equation represents the *incompressibility* of the flow and is linked to the role of the pressure p as a Lagrange multiplier.

Analyzing the incompressible Navier-Stokes or Euler equations presents considerable difficulties, especially in three dimensions. In the presence of viscosity, Leray famously constructed global weak solutions [69] which obey an energy inequality. However, it is of course unknown whether Leray's solutions remain smooth or exhibit finite-time blow-up. For the three-dimensional Euler equations, not even the existence of global weak solutions is known. Therefore, it is advantageous to seek a simplification of (NSE) which is more amenable to mathematical and numerical analysis but still accurately portrays the physical characteristics of atmospheric flows. This simplified system is the object of study in this dissertation and is known as the three-dimensional quasi-geostrophic system.

The main tool necessary to acquire the 3D QG system is the notion of an *asymptotic limit*. This notion is ubiquitous throughout fluid mechanics, with several notable examples being the inviscid limit, the compressible-incompressible limit, hydrodynamic limits of kinetic equations such as the Boltzmann equation, or homogenization problems. To study rotating fluids at high frequency, we must investigate the limit of the three-dimensional Navier-Stokes/Euler equations as the strength of the Coriolis force approaches infinity. This effect is quantified by the *Rossby number*, which is inversely related to the Coriolis force and is therefore quite small in our physical scenario. While detailed descriptions of this asymptotic analysis can be found in work of Bourgeois and Beale [7], Desjardins and Grenier [42], or in Chapter 4 of this work, we shall now give a rough outline.

For the time being, we ignore boundary conditions and focus only on the interior of the upper half-space, for which the effect of viscosity is negligible and the analysis is the same whether one begins with Navier-Stokes or Euler. The first step is to adimensionalize (NSE) according to the characteristic length scale L, velocity scale U, and time scale $\frac{L}{U}$ with the hopes of eliminating solutions which vary on a fast time scale. The Rossby number then appears in the equations and is given by

$$\epsilon = \frac{U}{f_0 L}.$$

The next step is to assume (mathematically this assumption is purely formal for now) that the velocity, pressure, and variation of the density with respect to a reference state satisfy

Hilbert expansions in powers of ϵ of the form

$$\vec{u} = \vec{u}_0 + \epsilon\vec{u}_1 + \epsilon^2\vec{u}_2 + O(\epsilon^2).$$

With this ansatz, one can send ϵ to zero and study the resulting balance. The zero-order equations are then (see Chapter 5 for a more thorough discussion)

$$v_0 = \partial_x p_0, \quad u_0 = -\partial_y p_0, \quad w_0 = 0, \quad \rho_0 = -\partial_z p_0.$$

The first three equations describe the *geostrophic balance*, which states that the zero-order geostrophic velocity is stratified and given by

$$(u_0, v_0, w_0) = \overline{\nabla}^{\perp} p_0 =: (-\partial_y p_0, \partial_x p_0, 0).$$

The last equation describes the *hydrostatic balance* and yields a better-posed problem than a purely hydrostatic assumption [7]. Continuing the analysis in a similar manner, one obtains the following equation at first order:

$$(\partial_t - \partial_y p_0 \partial_x + \partial_x p_0 \partial_y)\left(\partial_{xx} p_0 + \partial_{yy} p_0 + \partial_z\left(\lambda \partial_z p_0\right) + \beta_0 y\right) = 0. \tag{1.1}$$

Here $\lambda := (-\partial_z \rho_s)^{-1}$ comes from the density of the rest state, while β_0 is a parameter coming from a linear approximation of the effect of the latitude of the Earth. Introducing the notations for the stream function $\Psi = p_0$, the first order differential operators $\overline{\nabla} = (\partial_x, \partial_y, 0)$ and $\overline{\nabla}^{\perp} = (-\partial_y, \partial_x, 0)$, and the second order elliptic operator

$$\mathcal{L}(\cdot) = \partial_{xx} \cdot + \partial_{yy} \cdot + \partial_z\left(\lambda \partial_z \cdot\right)$$

($\partial_z \rho_s$ is assumed to be bounded below and above), we consolidate (1.1) into the following equation, representing the *conservation of potential vorticity* along material trajectories:

$$\left(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\right)\left(\mathcal{L}(\Psi) + \beta_0 y\right) = 0. \tag{1.2}$$

Note that (1.2) is not yet a closed system, as we must supplement it with boundary conditions at $z = 0$. In this work, we do not consider quasi-geostrophic flow in the full space $\mathbb{R}^3$, although interesting questions may be studied in this setting, such as the evolution of vortex patches [73]. In the next two subsections, we describe these boundary conditions and thus obtain the inviscid and viscous three-dimensional quasi-geostrophic models.

1.1.1 The Inviscid Model

A key parameter in the analysis of the Navier-Stokes equations is the Reynolds number. Supposing that $\mathcal{F}(\vec{u}) = -\mu\Delta\vec{u}$ in (NSE), the Reynolds number is then defined as

$$Re = \frac{UL}{\mu}, \tag{1.3}$$

where U and L are again characteristic velocity and length scales and μ is called the kinematic viscosity coefficient. The physical importance of the Reynolds number comes from the fact that the transition to turbulent behavior in a fluid is related to an increase in the Reynolds number. In his famous experiment [84], Reynolds pumped fluid through a pipe and showed that the laminar equilibrium becomes spontaneously unstable at sufficiently high Reynolds numbers. The Euler equations are a specific case of the Navier-Stokes equations with infinite Reynolds number. Applying this principle to quasi-geostrophic dynamics, one sets the viscous forces $\mathcal{F}(\vec{u})$ to be zero in the underlying (NSE). Noticing that $\rho_0 = -\partial_z p_0 = -\partial_z\Psi$ in the asymptotic limit of the Rossby number and recalling the continuity equation

$$\partial_t\rho + \vec{u}\cdot\nabla\rho = 0,$$

one can formally derive (again see Chapter 5 for more details, or the paper of Bourgeois and Beale [7]) that at $z = 0$,

$$\left(\partial_t + \overline{\nabla}^{\perp}\Psi|_{z=0}\cdot\overline{\nabla}\right)(-\partial_z\Psi|_{z=0}) = 0. \tag{1.4}$$

Using the notation $-\partial_z\Psi|_{z=0} = \partial_\nu\Psi$ and coupling this equation with the conservation of vorticity in (1.2) and an initial datum at $t = 0$ which we call Ψ_0, we have derived the three-dimensional inviscid quasi-geostrophic system $(QG)_I$:

$$(QG)_I \qquad \begin{cases} \left(\partial_t + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\right)(\mathcal{L}(\Psi) + \beta_0 y) = 0 \\ \left(\partial_t + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\right)(\partial_\nu\Psi) = 0 \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

Let us now describe the main mathematical characteristics of $(QG)_I$. The two nonlinear equations are in fact *advection* equations for the quantities $\partial_\nu\Psi$ and $\mathcal{L}(\Psi) + \beta_0 y$, with the advection or transport velocity being given by $\overline{\nabla}^{\perp}\Psi$. Formally, one can use the method of characteristics to assert that the pointwise values of the advected quantities are constant along the flow of the vector field $\overline{\nabla}^{\perp}\Psi$. Numerically, a simple time-stepping scheme for computing $(QG)_I$ would be as follows. Choose a temporal resolution scale Δt and suppose that the values of Ψ at time t are known. In order to predict the values of Ψ at time $t+\Delta t$, first consider the transport equations for the unknown functions $F : [t, t+\Delta t] \times \mathbb{R}^3_+ \to \mathbb{R}$ and $G : [t, t+\Delta t] \times \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} \left(\partial_t + \overline{\nabla}^{\perp}\Psi(t) \cdot \overline{\nabla}\right)(F + \beta_0 y) = 0 \\ \left(\partial_t + \overline{\nabla}^{\perp}\Psi(t)|_{z=0} \cdot \overline{\nabla}\right)(G) = 0 \\ F(t) = \mathcal{L}(\Psi)(t), \quad G(t) = \partial_\nu \Psi(t). \end{cases}$$

As these are transport equations on the time interval $[t, t+\Delta t]$ with constant velocity $\overline{\nabla}^{\perp}\Psi(t)$, solving for F and G at time $t+\Delta t$ is quite simple. Then, solving the elliptic Neumann problem in the upper half space

$$\begin{cases} \mathcal{L}(\Psi)(t+\Delta t) = F(t+\Delta t) \\ \partial_\nu \Psi(t+\Delta t) = G(t+\Delta t), \end{cases}$$

the values of Ψ at time $t+\Delta t$ have thus been computed. Repeating this process gives a global in time approximate solution of $(QG)_I$ and shows that $(QG)_I$ is a simple system to simulate.

The numerical scheme described above provides some insight into how to construct global in time weak solutions to $(QG)_I$. Solving an elliptic boundary value problem with Neumann condition is a compact operator, in the sense that for any bounded sequence of data $\{f_n\}$ for the elliptic operator and $\{g_n\}$ for the boundary data, the associated sequence of solutions $\{u_n\}$ contains a subsequence $\{u_{n_k}\}$ such that ∇u_{n_k} converges strongly. However, one can never hope to obtain strong convergence for $\{g_{n_k}\}$, indicating that passing to the limit at the boundary $z = 0$ for $(QG)_I$ is too much to hope for. It would therefore be desirable to obtain a formulation of $(QG)_I$ which *does not* rely on strong compactness at $z = 0$. An analogy should be drawn here between the stratified system $(QG)_I$ and the two-dimensional incompressible Euler equation in the full space $\mathbb{R}^2$. Letting $\vec{v} = (v^1, v^2)$ be the fluid velocity and $w := \overline{\nabla}^{\perp} \cdot v$ the associated vorticity (which is a scalar in two dimensions), it is well known that w satisfies

$$\partial_t w + v \cdot \overline{\nabla} w = 0.$$

The equations for $(QG)_I$ are analogous to the conservation of vorticity for 2D Euler, a first indication that quasi-geostrophic flow is better behaved mathematically than 3D Navier-Stokes or Euler.

The natural follow-up question to the preceding discussion therefore is: if $(QG)_I$ as stated is analogous to the vorticity formulation of Euler, can one state $(QG)_I$ in a way analogous to the velocity formulation of Euler? The answer is yes, and after setting the notation $\nabla_\lambda = (\partial_x, \partial_y, \lambda\partial_z)$, the reformulation can be written as [82]

$$\begin{cases} \left(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\right)(\nabla_\lambda \Psi) = \mathrm{curl}(Q) \\ \mathrm{curl}(Q) \cdot \nu = 0, \qquad z = 0 \\ \nabla_\lambda \Psi|_{t=0} = \nabla_\lambda \Psi_0. \end{cases}$$

In [82], Vasseur and Puel used the reformulation (albeit in a slighlty different format where a projection operator applied to the nonlinear term takes the place of the Lagrange multiplier $\mathrm{curl}(Q)$) to build global-in-time weak solutions to $(QG)_I$ for L^2 initial data. The reformulation then achieves the goal of viewing the system in such a way that passing to the limit does not require strong convergence at $z = 0$.

1.1.2 The Viscous Model

While inviscid models offer important insights into the behavior of turbulent flows, turbulence in physical experiments frequently stems from the presence of boundaries. In a fluid subject to frictional forces, the effects of viscosity are magnified greatly near the physical boundary. This is manifested mathematically in the different boundary conditions imposed for Navier-Stokes and Euler, namely the no-slip and no-penetration boundary conditions, respectively. Physically, one observes the presence of *boundary layers*, or thin layers of fluid near the walls for which viscous effects are significant. In quasi-geostrophic dynamics, these boundary layers are called Ekman layers [72], [42], [81] and result in the viscous term $\overline{\Delta}\Psi|_{z=0} := \partial_{xx} + \partial_{yy}\Psi$ being appended onto the transport equation for $\partial_\nu \Psi$ (we refer to Desjardins and Grenier [42] for a derivation)

$$(QG)_V \qquad \begin{cases} \left(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\right)(\mathcal{L}(\Psi) + \beta_0 y) = 0 \\ \left(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\right)(\partial_\nu \Psi) = \overline{\Delta}\Psi \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

While the dissipative term $\overline{\Delta}\Psi$ aids in the analysis at the boundary $z = 0$, note that while $\partial_\nu \Psi$ is advected, the diffusion occurs at the level of Ψ itself. In other words, one cannot simply treat the equation at $z = 0$ as a standard diffusive equation since the dissipative effects are not immediately visible at the level of $\partial_\nu \Psi$. Weak solutions to $(QG)_V$ were constructed by Desjardins and Grenier using classical methods [42]. We remark that since the construction of weak solutions due to Desjardins and Grenier does not require the reformulation in terms of $\nabla_\lambda \Psi$, it is therefore significantly simpler than the corresponding result of Vasseur and Puel [82] and precedes it by more than a decade.

1.2 A Special Case: The Two-Dimensional Surface Quasi-Geostrophic Equation

While 3D QG is already simpler than the primitive equations from which it is derived, a significant amount of mathematical research focuses on a further simplification known as the two-dimensional surface quasi-geostrophic equation (2D SQG). Suppose that $\lambda \equiv 1$ and $\beta_0 = 0$ so that the conservation of potential vorticity reads

$$\left(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\right) \Delta\Psi = 0.$$

Then since the values of $\Delta\Psi$ itself are advected by the fluid velocity, specifying a harmonic initial datum (i.e. $\Delta\Psi = 0$ in $\mathbb{R}^3_+$) should produce a solution which remains harmonic for each time $t > 0$. Furthermore, it is well-known that for a harmonic function u in the upper half space, the half-Laplacian $(-\overline{\Delta})^{\frac{1}{2}}$ acts as the *Dirichlet-to-Neumann* operator via the formula

$$-\partial_z u|_{z=0} = (-\overline{\Delta})^{\frac{1}{2}} u|_{z=0}. \tag{1.5}$$

Here $(-\overline{\Delta})^{\frac{1}{2}}$ can be defined in a number of ways which are known to be equivalent in $\mathbb{R}^2$, for example in terms of the Fourier multiplier $|\xi|$ (ignoring constants coming from the Fourier transform), or in terms of a singular integral kernel. We remark that a simple way to verify (1.5) is by recalling that the Fourier multiplier for the Laplace equation in the upper half space is $e^{-z|\xi|}$ and then calculating both sides of the equality directly. Then letting $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ be the vector of two-dimensional Riesz transforms (which can also be defined equivalently in terms of Fourier multipliers or singular integral kernels), one has in addition that

$$\overline{\nabla} u|_{z=0} = \mathcal{R}(-\overline{\Delta})^{\frac{1}{2}} u|_{z=0}.$$

Therefore, if Ψ is a solution to $(QG)_I$ which is harmonic at each time t, one can rewrite the transport equation for $\partial_\nu \Psi$ as

$$\partial_t((-\overline{\Delta})^{\frac{1}{2}}\Psi) + \mathcal{R}^{\perp}\left((-\overline{\Delta})^{\frac{1}{2}}\Psi\right) \cdot \overline{\nabla}((-\overline{\Delta})^{\frac{1}{2}}\Psi) = \mu\overline{\Delta}\Psi.$$

Here we take μ to be 0 or 1 depending on whether we are considering inviscid or viscous (QG), respectively. Adhering to the standard notations

$$\theta = -\partial_\nu \Psi, \qquad u = \overline{\nabla}^{\perp} \Psi|_{z=0}$$

and noticing that

$$-\overline{\Delta}\Psi|_{z=0} = (-\overline{\Delta})^{\frac{1}{2}} \circ (-\overline{\Delta})^{\frac{1}{2}} \Psi_{z=0} = (-\overline{\Delta})^{\frac{1}{2}}\theta,$$

we arrive at the familiar form of the 2D SQG equation

$$\partial_t \theta + u \cdot \overline{\nabla}\theta = -\mu(-\overline{\Delta})^{\frac{1}{2}}\theta. \tag{1.6}$$

When $\mu = 0$, the equation is known as *inviscid* SQG, while when $\mu > 0$ the equation is termed *critical* SQG. The moniker "critical" is justified by noticing that if θ is a solution to critical SQG, $\theta_s(t,x) := \theta(st, sx)$ is a solution as well. That is, dilating a solution according to the fractional parabolic scaling produces a new solution to the same equation for which the L^∞ norm remains unchanged. By solving Laplace's equation in the upper half space with Neumann data θ, any solution to 2D SQG should be a solution to 3D QG as well (see Chapter 3 for details). Furthermore, regularity estimates for Laplace's equation in the upper half space show that any interesting mathematical behavior must be occuring near the boundary $z = 0$. In other words, the dynamics of 3D QG in the absence of interior vorticity, i.e. $\Delta\Psi = 0$, can be encompassed by simply studying $\theta = \partial_\nu \Psi$.

Both inviscid and critical SQG have been the subject of extensive mathematical research (cf. [17], [28], [37], [63], [27], [83], [44], [50], [62], [9] and references therein), and many of the aforementioned works have inspired the work of this dissertation. In order to motivate the statements of the main results of this work in the next section, let us describe first their precedents in the setting of 2D SQG. We shall organize the results to correspond with the chronological writing of this dissertation (rather than the chronological order of the works themselves).

The Leray-Hopf solutions for critical SQG ($\theta \in L^\infty_t L^2_x \cap L^2_x \dot{H}^1_x$) are in fact globally C^∞ and unique. A number of proofs of this theorem have been offered; from Caffarelli and Vasseur [17] using the method of De Giorgi, Constantin and Vicol [28] and Constantin, Vicol, and Tarfulea [37] using a nonlinear maximum principle for the fractional Laplacian, Kiselev, Nazarov, and Volberg [63] using a nonlinear maximum principle for the modulus of continuity of the solution, and Kiselev and Nazarov [62] by transferring the evolution to a test function. Parts of the argument in Chapter 2 (along with many other works on non-local, nonlinear parabolic equations) are particularly inspired by the work of Caffarelli and Vasseur [17].

Weak solutions to inviscid 2D SQG were constructed by Resnick [83] by exploiting a special commutator structure in the nonlinearity which gives weak continuity in L^2 (weak solutions to critical 2D SQG can be constructed by classical methods, for example Galerkin approximation, and are smooth as noted above). Resnick's result was extended to initial data $\theta \in L^p$ for $p > \frac{4}{3}$ by Marchand [71]. Among other results, Chapter 3 includes arguments combining the techniques of Marchand with the reformulation of Vasseur and Puel.

A version of 2D SQG on bounded domains $\Omega \subset \mathbb{R}^2$ has been considered by Constantin et. al. in [33], [32], [36], [36], and [34]. Considering 3D QG on cylindrical domains reveals subleties in boundary conditions that are not visible in the 2D equation and which we shall explore in Chapter 4.

Finally, weak solutions to both critical and inviscid 2D SQG were shown to be nonunique and capable of achieving any smooth energy profile by Buckmaster, Shkoller and Vicol [9] using the method of convex integration. However, the main difficulties in proving nonuniqueness for 3D QG are decidedly different than the difficulties present in 2D SQG, and energy-dissipative solutions for for 2D SQG are fundamentally different than dissipative solutions to 3D QG, as we will show in Chapter 5.

1.3 Main Results and Outline of the Dissertation

The results in this dissertation are separated into four chapters, containing essentially the results of the papers [79], [77], [76], and [78]. Technical details and notations pertaining to the results in each chapter are collected in corresponding appendices. We shall now highlight the main theorems, their physical significance and motivation, and the main ideas from the proofs. For the sake of the clarity of this introduction, the statements of the theorems have been abbreviated slightly from the more general (and precise) statements contained in the chapters. We begin with the content of Chapter 2.

Theorem 2.1 (N.-Vasseur [79]). *Consider the viscous system $(QG)_V$ posed in the upper half space $\mathbb{R}^3_+$*

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}(\Delta\Psi) = 0 & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}(\partial_\nu\Psi) = \overline{\Delta}\Psi & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \Psi(0, z, x) = \Psi_0(z, x) & t = 0, \quad z \geq 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

Let the initial data $\nabla\Psi_0$ be a smooth (C^∞) function. Then there exists a unique, classical, $C^\infty_{t,x}$ solution Ψ to $(QG)_V$ which is stable with respect to perturbations of the initial data.

Physically, this theorem proves that the Ekman pumping term $\overline{\Delta}\Psi$ *stabilizes* the flow. The corresponding mathematical statement is the global regularity and stability of solutions with respect to initial data which may be arbitrarily large. To prove Theorem 2.1, we combined several techniques coming from elliptic regularity theory, harmonic analysis, and fluid equations. The first step occupies Section 2.3 and uses the De Giorgi method in the style of Caffarelli and Vasseur [17] to show that $\partial_\nu\Psi$ is Hölder continuous. Then in Section 2.4, we bootstrap the Hölder regularity of $\partial_z\Psi$ up to the critical Besov space $\dot{B}^1_{\infty,\infty}(\mathbb{R}^2)$ using a rather delicate combination of potential theory and Littlewood-Paley techniques (we refer to the work of Dong and Pavlovic [44] for a related result). Finally, the proof of Theorem 2.1 is completed in Section 2.5 using an analogue of the Beale-Kato-Majda criterion [4].

The remaining results all pertain to the inviscid system. Chapter 2 follows the paper [77] and ascertains the minimal properties one must impose on an initial datum to ensure that the inviscid system is still relevant. Mathematically, we therefore state an existence theorem for a notion of weak solution which is defined in terms of the reformulated problem (cf. Definition 3.1.1 and [82]), and which will be sufficiently general to unify the various mathematical representations of the inviscid system.

Theorem 3.1 (N. [77]). *Consider the inviscid system $(QG)_I$ posed in the upper half space $\mathbb{R}^3_+$*

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}(\Delta\Psi) = 0 & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}(\partial_\nu\Psi) = 0 & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \Psi(0, z, x) = \Psi_0(z, x) & t = 0, \quad z \geq 0, \quad x = (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

Let the initial data $\nabla\Psi_0$ have vorticity $\Delta\Psi_0 \in L^q\left(\mathbb{R}^3_+\right)$ and Neumann condition $\partial_\nu\Psi_0 \in L^p\left(\mathbb{R}^2\right)$ for p, q sufficiently large. Then there exists a solution Ψ satisfying the reformulation in a weak sense for which

$$\|\Delta\Psi(t)\|_{L^q} \leq \|\Delta\Psi_0\|_{L^q}, \qquad \|\partial_\nu\Psi(t)\|_{L^p} \leq \|\partial_\nu\Psi_0\|_{L^p}.$$

The following theorem then provides the link between our existence result and the previously utilized perspectives on $(QG)_I$ by asserting that the framework of the reformulated problem is sufficiently general to encompass the other theories. We state the theorem rather loosely for now; the precise statement is contained in Chapter 3.

Theorem 3.2 (N. [77]). *If the initial data Ψ_0 has sufficiently integrable vorticity $\Delta\Psi_0$ and Neumann condition $\partial_\nu\Psi_0$, weak solutions to the reformulated problem and weak solutions to $(QG)_I$ defined classically are equivalent. If in addition $\Delta\Psi_0 = 0$, then weak solutions to (SQG) defined by Marchand and weak solutions to the reformulated problem are also equivalent.*

In Chapter 4, we present a joint work with Vasseur [76], in which one considers $(QG)_I$ posed on a bounded domain of the form $\Omega \times [0, h]$ for $\Omega \subset \mathbb{R}^2$ a smooth, bounded set. The underlying goal of our work is to derive the physically natural model for an inviscid, stratified flow in the presence of nontrivial lateral boundaries. The result was a new mathematical model in which the advection of the potential vorticity and Neumann data is coupled with lateral boundary conditions which then close the system. In Section 4.2, we present a formal derivation from the primitive equations which asserts that a solution Ψ must satisfy the following lateral boundary conditions.

1. There exists a function $c(t,z)$ such that $\Psi(t,x,y,z)|_{\partial\Omega\times[0,h]} = c(t,z)$.

2. The time derivative of the "average Neumann condition" on the lateral boundary vanishes; that is, if ν_s is the unit normal on $\partial\Omega \times [0,h]$,
$$\frac{d}{dt}\fint_{\partial\Omega\times\{z\}} \nabla\Psi \cdot \nu_s = 0.$$

Note that we are *not* allowed to specify $c(t,z)$. In particular, we cannot choose Dirichlet boundary data for convenience, implying that our solutions cannot coincide with those of Constantin et. al. for SQG on bounded domains. In some sense, condition (1) leaves one degree of freedom unspecified at each time t in the form in the form of a function of z defined on $[0,h]$. However, the second condition balances this out by requiring the specification of a function $j(z)$ which dictates the "average Neumann condition" at each time t.

Before building weak solutions to the model we derived, we prove in Section 4.3 an elliptic regularity theorem for a stationary problem incorporating the nonstandard boundary conditions (1) and (2). Here the interaction of the geometry of the domain $\Omega \times [0,h]$ with the boundary conditions becomes crucial, as elliptic regularity in Lipschitz domains is not always available. Nevertheless, our elliptic regularity theorem is strong enough to prove in Section 3.3 the following theorem concerning the existence of weak solutions to the system we derived.

Theorem 4.1 (N.-Vasseur [76]). *Given initial data for the initial vorticity $\Delta\Psi_0$, Neumann condition $\partial_\nu\Psi_0$, and "average Neumann condition"*
$$\fint_{\partial\Omega\times\{z\}} \nabla\Psi_0 \cdot \nu_s,$$
there exists a global weak solution to $(QG)_I$ posed on $\Omega \times [0,h]$ satisfying the boundary conditions (1) and (2).

The final chapter of the dissertation concerns the nonuniqueness and dissipation of kinetic energy of weak solutions to $(QG)_I$. The physical motivation for the consideration of such weak solutions comes from what is sometimes called Kolmogorov's *zeroth law of turbulence*, an assumption which undergirds the famous K41 theory (we refer to Kolmogorov's classic work [65] as well as the survey paper [15] by Buckmaster and Vicol and references therein for mathematical perspectives). Kolmogorov's zeroth law asserts that energy may dissipate anomalously in the limit of zero viscosity. Mathematically, one then seeks rigidity results, which specify conditions under which the energy is preserved, and flexiblity results, which specify the classes of weak solutions in which one may expect anomalous dissipation of energy. In order to allow for the application of Fourier analytic tools, it is advantageous to consider the periodic domain $\mathbb{T}^3$ (of course physical boundaries are crucial to a full understanding of turbulence but present considerable mathematical difficulties). We begin by proving the following rigidity theorem for dissipative weak solutions (of the reformulated problem).

Theorem 5.1 (N. [77]). *Let $\nabla\Psi$ be a weak solution such that*

$$\nabla\Psi \in C\left([0,T);L^2(\mathbb{T}^3)\right) \cap L^3\left([0,T)\times(0,2\pi);C^\alpha(\mathbb{T}^2)\right)$$

for some $\alpha > \frac{1}{3}$. Then $\|\nabla\Psi(t)\|_{L^2(\mathbb{T}^3)} = \|\nabla\Psi_0\|_{L^2(\mathbb{T}^3)}$ for $t \in [0,T)$.

The proof of this theorem is relatively short using Littlewood-Paley techniques following a similar result for the Euler equations proved by Constantin, E, and Titi [31]. The remainder of the chapter is then devoted to a presentation of the work [78], in which we proved the following flexibility result.

Theorem 5.2 (N. [78]). *Let $e:\mathbb{R}\to[0,\infty)$ be a smooth, compactly supported function and $\zeta \in \left(0,\frac{1}{5}\right)$. Then there exists a weak solution $\nabla\Psi \in C^\zeta\left(\mathbb{R}\times\mathbb{T}^3\right)$ of the reformulated problem such that*

$$\int_{\mathbb{T}^3} |\nabla\Psi(t,x,y,z)|^2\,dx\,dy\,dz = e(t).$$

This theorem shows that below a certain Hölder regularity threshold, weak solutions are nonunique and may dissipate the total kinetic energy. The proof of this theorem utilizes the modern techniques of convex integration pioneered by De Lellis and Szekelyhidi [68]. Roughly speaking, the convex integration scheme is an inductive process via which the Littlewood-Paley projections (for a sequence of frequency shells which, for technical reasons, grow super-exponentially) of a dissipative weak solution are specified step by step. In each step, an approximate solution $\nabla\Psi_q$ is constructed which is compactly supported in frequency in a ball of radius λ_q and solves $(QG)_I$ up to a small error of size δ_q. The main work of the argument then consists of showing that this process can be continued inductively while sending λ_q to ∞ and δ_q to 0, thus obtaining a dissipative weak solution in the limit. Nonuniqueness follows by simply noticing that the constant function 0 is a steady solution which will coincide with a convex integration solution on the time interval where $e(t) = 0$ but not where $e(t) > 0$.

Chapter 2

Uniqueness of Global Smooth Solutions for the Viscous Model

2.1 Overview

In this chapter [1], we consider the viscous 3D quasi-geostrophic system $(QG)_V$, which can be stated as the following set of equations imposed upon the stream function $\Psi : [0,\infty) \times \mathbb{R}^3_+ \to \mathbb{R}$:

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\Delta\Psi) = 0 & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\partial_\nu\Psi) = \overline{\Delta}\Psi & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \Psi(0, z, x) = \Psi_0(z, x) & t = 0, \quad z \geq 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \end{cases} \qquad (\mathrm{QG})_V \ .$$

Note that for the sake of mathematical expediency, we have simplified the equation for the conservation of potential vorticity by setting $\lambda(z) = 1$, which turns the elliptic operator $\mathcal{L}$ into the standard Laplacian. We have also eliminated the β-plane approximation, which gave rise to the term $\beta_0 y$. Throughout this chapter, we employ the convention that the vertical component is the first component of any vector in $\mathbb{R}^3_+$. We set the notations

$$\overline{\nabla}\Psi = (0, \partial_{x_1}\Psi, \partial_{x_2}\Psi),$$

and

$$\overline{\Delta}\Psi = \partial_{x_1x_1}\Psi + \partial_{x_2x_2}\Psi.$$

As usual, the velocity field for the stratified flow is given by

$$\overline{\nabla}^\perp\Psi = (0, -\partial_{x_2}\Psi, \partial_{x_1}\Psi).$$

[1]The contents of this chapter are based on the joint work of the author with Alexis Vasseur "Global in Time Classical Solutions for the 3D Quasi-Geostrophic System for Large Initial Data, *Communications in Mathematical Physics*, 358(1):237-267, Nov. 2017. Both authors contributed equally to this work."

At the boundary $z = 0$, $\partial_\nu \Psi$ is a function of x and t only and denotes the Neumann condition

$$\partial_\nu \Psi(t, x) = -\partial_z \Psi(t, 0, x).$$

Of course $\Delta\Psi = \partial_{zz}\Psi + \partial_{x_1x_1}\Psi + \partial_{x_2x_2}\Psi$ is the usual Laplacian.

The viscous 3D quasi-geostrophic system has been used to model large-scale, stratified oceanic and atmospheric circulation exhibiting geostrophic balance (see Pedlosky [81]). Chemin [18] considered the convergence of solutions to the primitive equations to a solution of the quasi-geostrophic equation in the asymptotic limit of the Rossby number. We recall that rigorous derivations of the equations were carried out by Beale and Bourgeois [7] in the absence of the boundary layer and Desjardins and Grenier [42] with the inclusion of the boundary layer. Much of the difficulty in the analysis in fact stems from the boundary layer. Taking advantage of the viscous term on the boundary, Desjardins and Grenier [42] constructed global weak solutions.

This chapter is dedicated to a proof of the following well-posedness result for $(QG)_V$ and follows the work of the author and Vasseur [79].

Theorem 2.1. *Let the initial data $\nabla\Psi_0 \in H^s(\mathbb{R}^3_+)$ for some $s \geq 3$. Then there exists a unique classical solution Ψ to $(\mathcal{QG})$ satisfying the following: for all $T > 0$, there exists $C(T, s)$ such that for all $t \leq T$, $||\nabla\Psi(t, \cdot)||_{H^s(\mathbb{R}^3_+)} \leq C(T, s)$. In addition, if the initial data $\nabla\Psi_0 \in H^s(\mathbb{R}^3_+)$ for all s, then for all T, $\Psi \in C^\infty([0, T] \times \mathbb{R}^3_+)$.*

The bulk of the proof is centered around verifying a version of the Beale-Kato-Majda criterion [4]. In the context of the Euler equations, the Beale-Kato-Majda criterion states that a smooth solution u to the 3D Euler equations can blow up at time T if and only if

$$\int_0^T \|\nabla \times u(s)\|_{L^\infty}\, ds = \infty.$$

The proof uses the Biot-Savart law to show that the vorticity $\nabla \times u$ controls a norm on ∇u which is weaker than L^∞ but strong enough to propagate H^s norms of the initial data. In our context, we will show that for each $z \in [0, \infty)$, the Besov norm $\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2)$ of the velocity field $\overline{\nabla}^\perp\Psi(z)$ remains bounded in time. In the literature, this space is referred to as the Zygmund class or $C^{1,\star}$. The texts of Stein [90] and Grafakos [53] include thorough expositions of the essential theory, while Chemin [19] and Bahouri, Chemin, and Danchin [3] have detailed the application of the Zygmund class to the study of wide classes of PDE's, particularly the incompressible Euler equations. For us, the most useful property of Besov spaces will be an inequality which controls the L^∞ norm of the velocity field $\overline{\nabla}^\perp\Psi$ by the $\mathring{B}^0_{\infty,\infty}$ Besov norm, a lower Sobolev norm, and a logarithm of a higher Sobolev norm.

In the proof, we first decompose the solution $\Psi = \Psi_1 + \Psi_2$ into two components as follows:

$$\begin{cases} \Delta\Psi_1 = 0 \\ \partial_\nu\Psi_1 = \partial_\nu\Psi \end{cases} \qquad \begin{cases} \Delta\Psi_2 = \Delta\Psi \\ \partial_\nu\Psi_2 = 0. \end{cases}$$

Since Ψ_2 encodes the effect of the vorticity but not the boundary condition and the L^∞ norm of the vorticity is conserved in time, one can intuit that $\overline{\nabla}^\perp\Psi_2$ already satisfies a satisfactory Beale-Kato-Majda type estimate. Thus Ψ_1 is the problematic term since it contains the boundary condition. We will find that $\partial_\nu\Psi_1$ satisfies an equation resembling critical 2D SQG, with an adjustment to the drift term and a forcing term appearing due to the presence of non-zero interior vorticity. To show that $\partial_\nu\Psi_1$ is Hölder continuous, we utilize the De Giorgi technique following [17] and [96] (see also Friedlander and Vicol [51] for an application to active scalar equations). We then improve the regularity of $\partial_\nu\Psi_1$ using Littlewood-Paley techniques and potential theory to bootstrap (see [17], Constantin and Wu [29], and Dong and Pavlović [44]). Due to the fact that $\partial_x\Psi_1, \partial_y\Psi_1$ are related to $\partial_z\Psi_1$ via the Riesz transforms, the $\mathring{B}^1_{\infty,\infty}$ norm on the velocity field $\overline{\nabla}^\perp\Psi_1$ is then preserved. From there, we can prove propagation of regularity.

2.2 Notations and a Priori Estimates

In this chapter, we consider functions defined on $\mathbb{R}^2$ or $\mathbb{R}^3_+ = [0,\infty) \times \mathbb{R}^2$. It will be convenient to keep track of when functions are being differentiated in $x = (x_1, x_2)$ only. For that reason, and also to emphasize when we are considering functions defined on $\mathbb{R}^2$, we employ the following notations.

Definition 2.2.1. *Let f be a real-valued function defined on $\mathbb{R}^3_+$. Put $\overline{\Delta} f = \partial_{x_1x_1}f + \partial_{x_2x_2}f$ and $\overline{\nabla} f = (0, \partial_{x_1}f, \partial_{x_2}f)$. Let $((-\overline{\Delta})^\alpha f)\hat{}\,(z,\xi) = \hat{f}(z,\xi)\cdot|\xi|^{2\alpha}$, where the Fourier transform is* *being taken in x only for each fixed z (ignoring constants coming from the Fourier transform). For a partial differential operator with multi-index $\alpha = (\alpha_1, \alpha_2)$, $\overline{D}^\alpha f$ denotes differentation in the flat variables (x_1, x_2). When f is only defined on $\mathbb{R}^2$, we will use the above symbols to denote the usual differential operators.*

We now state a local existence theorem and the necessary *a priori* estimates. For a proof of the following local existence theorem, one can employ the standard semigroup approach found in Kato [61].

Proposition 2.2.1. *For any initial data $\nabla\Psi_0 \in H^3(\mathbb{R}^3_+)$ for $(\mathcal{QG})$, there exists a time interval $[0, \bar{T}]$, where $\bar{T}$ depends only on the size of $||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$, such that $(\mathcal{QG})$ has a solution $\nabla\Psi \in L^\infty([0,T]; H^3(\mathbb{R}^3_+))$.*

The following proposition contains the a priori estimates which we shall use to prove global existence. Each estimate depends only on the size of the norm of the initial data $||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$. The definitions of the somewhat large variety of various function spaces and operators are contained in Appendix A.

Proposition 2.2.2. *Let $\nabla\Psi \in L^\infty([0,T]; H^3(\mathbb{R}^3_+))$ be a smooth solution to (QG) on the interval $[0,T]$. Then there exists a universal C independent of T and Ψ such that Ψ satisfies the following for all $t \in [0,T]$:*

1. $\frac{1}{2}||\nabla\Psi(t)||^2_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}\Psi(t)|_{z=0}||^2_{L^2([0,t];L^2(\mathbb{R}^2))} \le ||\nabla\Psi_0||^2_{L^2(\mathbb{R}^3_+)}$.
2. *For all* $p \in [2,\infty]$, $||\Delta\Psi(t)||_{L^p(\mathbb{R}^3_+)} = ||\Delta\Psi_0||_{L^p(\mathbb{R}^3_+)} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.
3. $||(-\overline{\Delta})^{\frac{3}{4}}\Psi_2(t)|_{z=0}||_{L^2(\mathbb{R}^2)} \le ||\Delta\Psi_0||_{L^2(\mathbb{R}^3_+)} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.
4. *For* $z = z_0 \ge 0$, $||\nabla\Psi_2(t)|_{z=z_0}||_{\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2)} \le ||\Delta\Psi_0||_{L^\infty(\mathbb{R}^3_+)} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.
5. $||(-\overline{\Delta})^{\frac{3}{4}}\Psi_2(t)|_{z=0}||_{C^{\frac{1}{2}}(\mathbb{R}^2)} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.
6. $||\partial_\nu\Psi(t)||_{L^2(\mathbb{R}^2)} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}(1+t)$.
7. *For* $p \in [4,\infty]$ *and* $z_0 \ge 0$, $||\nabla\Psi_2(t)|_{z=z_0}||_{L^p} \le C||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.

Proof. 1. We multiply (QG_∇) by $\nabla\Psi$ and integrate. By the properties of the projection operator $\mathbb{P}_\nabla$ and the divergence-free and stratified nature of the flow,

$$\int_{\mathbb{R}^3_+} \mathbb{P}_\nabla(\overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\nabla\Psi)) \cdot \nabla\Psi = \int_{\mathbb{R}^3_+} \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\nabla\Psi) \cdot \nabla\Psi = 0.$$

Therefore we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3_+} |\nabla\Psi|^2 = \int_{\mathbb{R}^3_+} \nabla\Psi\nabla F &= -\int_{\mathbb{R}^3_+} \Psi\Delta F + \int_{\mathbb{R}^2} \Psi|_{z=0}\partial_\nu F \\ &= \int_{\mathbb{R}^2} \Psi|_{z=0}\overline{\Delta}\Psi|_{z=0} \\ &= -\int_{\mathbb{R}^2} |\overline{\nabla}\Psi|_{z=0}|^2\end{aligned}$$

Integrating in time then gives the claim.

2. The estimate follows immediately from the transport equation for $\Delta\Psi$, the divergence free property of the flow, and Sobolev embedding.

3. We define $\tilde{\Psi}_2(z,x) = \Psi_2(|z|,x)$. Note that $\Delta\tilde{\Psi}_2(z,x) = \Delta\Psi_2(|z|,x)$ and $\nabla\tilde{\Psi}_2(z,x) = -\nabla\Psi_2(|z|,x)$. Applying the Riesz transforms to $\Delta\tilde{\Psi}_2$ and using (2) and parts (2) and (3) of Proposition A.0.3 shows that $\nabla^2\tilde{\Psi}_2 \in L^2(\mathbb{R}^3)$, and therefore $\nabla^2\Psi_2 \in L^2(\mathbb{R}^3_+)$. Applying Lemma A.0.1 shows that

$$\nabla\Psi_2|_{z=0} \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2) \tag{2.1}$$

and it follows immediately from the Fourier characterization of $\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ that $\partial_x\Psi_2, \partial_y\Psi_2 \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ implies $(\overline{\Delta})^{\frac{3}{4}}\Psi_2|_{z=0} \in L^2(\mathbb{R}^2)$.

4. We use again that $\Delta\tilde{\Psi}_2(z,x) = \Delta\Psi_2(|z|,x)$ together with parts (2) and (4) of Proposition A.0.3 to obtain $\nabla\tilde{\Psi}_2|_{z=z_0} \in \mathring{B}^1_{\infty,\infty}(\mathbb{R}^3_+)$. Using part (3) of Proposition A.0.4 with $s=1$, $n=3$, and $k=2$ and recalling that $\nabla\tilde{\Psi}_2(z,x) = \nabla\Psi_2(|z|,x)$, we have $\nabla\Psi_2|_{z=z_0} \in \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2)$.

5. To obtain (5), we can use (3) and (4). Using (4), Proposition A.0.4, and the Riesz transform shows that $-(\overline{\Delta})^{\frac{3}{4}}\Psi_2 \in \mathring{C}^{\frac{1}{2}}(\mathbb{R}^2)$. To show that $-(\overline{\Delta})^{\frac{3}{4}}\Psi_2$ actually belongs to the inhomogenous space $C^{\frac{1}{2}}$, we must show that $-(\overline{\Delta})^{\frac{3}{4}}\Psi_2 \in L^\infty(\mathbb{R}^2)$. This follows from (3) and the $\mathring{C}^{\frac{1}{2}}$ bound.

6. We take the equation on the boundary $z = 0$, multiply by $\partial_\nu \Psi(t)$, and apply (3), yielding

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^2} |\partial_\nu \Psi(t)|^2 &= \int_{\mathbb{R}^2} \overline{\Delta}\Psi(t)\partial_\nu \Psi(t) \\
&= \int_{\mathbb{R}^2} \overline{\Delta}\Psi_1(t)\partial_\nu \Psi_1(t) + \int_{\mathbb{R}^2} \overline{\Delta}\Psi_2(t)\partial_\nu \Psi_1(t) \\
&= -\int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{2}}\partial_\nu \Psi_1(t)|^2 + \int_{\mathbb{R}^2} \overline{\Delta}\Psi_2(t)\partial_\nu \Psi_1(t) \\
&\leq -||\partial_\nu \Psi_1(t)||^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + ||\overline{\Delta}\Psi_2(t)||_{\mathring{H}^{-\frac{1}{2}}(\mathbb{R}^2)}||\partial_\nu \Psi_1(t)||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \\
&\leq -||\partial_\nu \Psi_1(t)||^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + ||(-\overline{\Delta})^{\frac{3}{4}}\Psi_2(t)||^2_{L^2(\mathbb{R}^2)} + ||\partial_\nu \Psi_1(t)||^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \\
&\leq C||\nabla \Psi_0||_{H^3(\mathbb{R}^3_+)}
\end{aligned}$$

Integrating in time finishes the proof.

7. The estimate follows from (2.1), Sobolev embedding, (4), and interpolation.

□

Finally, let us remark that constants C may change from line to line; if we wish to keep track of dependencies, we will write $C(\cdot)$.

2.3 The Hölder Estimate for $\partial_\nu \Psi$

Let us examine $\partial_\nu \Psi_1 = \partial_\nu \Psi$. We have that $\partial_\nu \Psi_1$ satisfies the equation

$$\partial_t(\partial_\nu \Psi_1) + \overline{\nabla}^\perp \Psi|_{z=0} \cdot \overline{\nabla}(\partial_\nu \Psi_1) + (-\overline{\Delta})^{\frac{1}{2}}(\partial_\nu \Psi_1) = \overline{\Delta}\Psi_2|_{z=0}.$$

In this section we prove the following regularity estimate on $\partial_\nu \Psi$.

Lemma 2.3.1 (Hölder Estimate)**.** *If $\nabla \Psi \in L^\infty([0,T]; H^3(\mathbb{R}^3_+))$ solves (QG) on $[0,T]$, there exists $r > 0$, $C > 0$ depending only on $||\nabla \Psi_0||_{H^3(\mathbb{R}^3_+)}$ such that the following holds. The solution $\partial_\nu \Psi$ to the boundary equation*

$$\partial_t(\partial_\nu \Psi) + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\partial_\nu \Psi) = \overline{\Delta}\Psi$$

satisfies $\partial_\nu \Psi \in C^r([0,T] \times \mathbb{R}^2)$ with $||\partial_\nu \Psi||_{C^r([0,T]\times\mathbb{R}^2)} < C$.

The steps of the De Giorgi argument are written for equations of the type

$$\partial_t \theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f.$$

We will apply the De Giorgi lemmas to $\theta = \partial_\nu \Psi$ to obtain Lemma 2.3.1. Estimates for θ, u, and f will come from Proposition 2.2.2; in particular, they will only depend on $||\nabla \Psi_0||_{H^3(\mathbb{R}^3_+)}$. We begin with the first De Giorgi lemma which will give an estimate on $||\theta||_{L^\infty([0,T]\times\mathbb{R}^2)}$. Let us remark that all parts of the De Giorgi argument will be applied on the interval for which (QG) has a solution $\nabla\Psi \in L^\infty([0,T]; H^3(\mathbb{R}^3_+))$. From the trace, we have $\partial_\nu \Psi, \overline{\nabla}^\perp \Psi \in L^\infty([0,T]; H^{\frac{5}{2}}(\mathbb{R}^3_+))$, which justifies the calculations.

2.3.1 From L^2 to L^∞

We begin with a technical proposition which we shall use several times to estimate the forcing term f.

Proposition 2.3.2. *Suppose that* $||-(\bar{\Delta})^{-\frac{1}{4}} g(t,x)||_{L^\infty([-2,0];C^{\frac{1}{2}}(\mathbb{R}^2))} \leq M$*, and* $\omega(t,x)$ *satisfies the following:*

1. $\omega(t,x) \in L^\infty([-2,0]; L^2 \cap L^1(\mathbb{R}^2)) \cap L^2([-2,0]; \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2))$
2. $\omega(t,x) \geq 0$ *for* $(t,x) \in [-2,0] \times \mathbb{R}^2$
3. *For each time* t*,* $|\{(t,x) : \omega(t,x) > 0\}| < \infty$

Then

$$\int_{\mathbb{R}^2} g(t,x)\omega(t,x) \leq C(M)\left(\int_{\mathbb{R}^2} \omega(t,x)\,dx + \int_{\mathbb{R}^2} \mathcal{X}_{\{\omega(t,x)>0\}}\,dx\right) + \frac{1}{2}||\omega(t,\cdot)||^2_{\mathring{H}^{\frac{1}{2}}}$$

Proof. Put $h(t,x) = (-\overline{\Delta})^{-\frac{1}{4}} g(t,x)$; we then have

$$\begin{aligned}
I = \int_{\mathbb{R}^2} g(t,x)\omega(t,x)\,dx &= \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{8}} h(t,x)(-\overline{\Delta})^{\frac{1}{8}}\omega(t,x)\,dx \\
&\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|h(t,x)-h(t,y)||\omega(t,x)-\omega(t,y)|}{|x-y|^{\frac{5}{2}}}\,dx\,dy \\
&= \iint_{|x-y|\leq 1} \frac{|h(t,x)-h(t,y)||\omega(t,x)-\omega(t,y)|}{|x-y|^{\frac{5}{2}}}\,dx\,dy \\
&+ \iint_{|x-y|>1} \frac{|h(t,x)-h(t,y)||\omega(t,x)-\omega(t,y)|}{|x-y|^{\frac{5}{2}}}\,dx\,dy \\
&= I_1 + I_2.
\end{aligned}$$

We begin by estimating I_2. Since $h(t,x) \in L^\infty(C^{\frac{1}{2}})$, we have that

$$\begin{aligned}
I_2 \leq& \iint_{|x-y|>1} \frac{2M|\omega(t,x)-\omega(t,y)|}{|x-y|^{\frac{5}{2}}}\,dx\,dy \\
\leq& 4M \iint_{|x-y|>1} \frac{\omega(t,x)}{|x-y|^{\frac{5}{2}}}\,dx\,dy \\
\leq& 4M \int_1^\infty r^{-\frac{3}{2}}\,dr \int_{\mathbb{R}^2} \omega(t,x)\,dx \\
\leq& 4M \int_{\mathbb{R}^2} \omega(t,x)\,dx
\end{aligned}$$

We must now estimate I_1. Using the symmetry in x and y and the fact that

$$|\omega(t,x)-\omega(t,y)| \leq (\mathcal{X}_{\{\omega(t,x)>0\}} + \mathcal{X}_{\{\omega(t,y)>0\}})|\omega(t,x)-\omega(t,y)|$$

we have

$$I_1 \leq 2\iint_{|x-y|\leq 1} \mathcal{X}_{\{\omega(x)>0\}} \frac{|h(t,x)-h(t,y)||\omega(t,x)-\omega(t,y)|}{|x-y|^{\frac{5}{2}}}\,dx\,dy.$$

By Cauchy's inequality, we have

$$\begin{aligned}
I_1 \leq& 4\iint_{|x-y|\leq 1} \mathcal{X}_{\{\omega(x)>0\}} \frac{|h(t,x)-h(t,y)|^2}{|x-y|^2}\,dx\,dy \\
&+ \frac{1}{4}\iint_{|x-y|\leq 1} \frac{|\omega(t,x)-\omega(t,y)|^2}{|x-y|^3}\,dx\,dy.
\end{aligned}$$

Using the $L^\infty(C^{\frac{1}{2}})$ regularity of h, we have that

$$\begin{aligned}
4\iint_{|x-y|\le 1} \mathcal{X}_{\{\omega(t,x)>0\}} \frac{|h(t,x)-h(t,y)|^2}{|x-y|^2}\,dx\,dy \\
&\le 4\iint_{|x-y|\le 1} \mathcal{X}_{\{\omega(t,x)>0\}} \frac{M^2|x-y|}{|x-y|^2}\,dx\,dy \\
&\le 4M^2\int_0^1 dr\int_{\mathbb{R}^2} \mathcal{X}_{\{\omega(t,x)>0\}}\,dx \\
&\le 4M^2\int_{\mathbb{R}^2} \mathcal{X}_{\{\omega(t,x)>0\}}\,dx
\end{aligned}$$

and

$$\frac{1}{4}\iint_{|x-y|\le 1} \frac{|\omega(t,x)-\omega(t,y)|^2}{|x-y|^3}\,dx\,dy \le \frac{1}{2}||\omega(t,\cdot)||^2_{\mathring{H}^{\frac{1}{2}}},$$

concluding the proof. □

Lemma 2.3.3 (Global L^∞ bound)**.** *For any $M>0$, there exists $L>0$ such that the following holds. Let $\theta \in L^\infty([-2,0];H^{\frac{5}{2}}(\mathbb{R}^2))$ be a solution to*

$$\partial_t\theta + u\cdot\overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f$$

with

$$||\theta||_{L^\infty([-2,0];L^2(\mathbb{R}^2))} + ||(-\overline{\Delta})^{-\frac{1}{4}}f||_{L^\infty([-2,0];C^{\frac{1}{2}}(\mathbb{R}^2))} < M$$

and $\operatorname{div} u = 0$*. Then $\theta(t,x)\le L$ for $(t,x)\in[-1,0]\times\mathbb{R}^2$.*

Proof. The main tool in showing the L^∞ bound is an energy inequality, which we now derive. Fix a constant $c>0$, and define $\theta_c := (\theta-c)_+$. Multiplying the equation by θ_c, integrating in space, and using that the drift is divergence-free, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\theta_c^2(t,x)\,dx + \int_{\mathbb{R}^2}\theta_c(t,x)(-\overline{\Delta})^{\frac{1}{2}}\theta(t,x)\,dx = \int_{\mathbb{R}^2} f(t,x)\theta_c(t,x)\,dx.$$

Making use of a pointwise estimate of Córdoba and Córdoba [38], we have that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\theta_c^2(t,x)\,dx + \int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}\theta_c(t,x)|^2\,dx \le \int_{\mathbb{R}^2} f(t,x)\theta_c(t,x)\,dx. \tag{2.2}$$

Applying Proposition 2.3.2 with $g = f$, $\theta_c = \omega$, we obtain the energy inequality

$$\frac{d}{dt}\int_{\mathbb{R}^2}\theta_c^2(t,x)\,dx + \int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}\theta_c(t,x)|^2\,dx \le C(M)\left(\int_{\mathbb{R}^2}\theta_c(t,x)\,dx + \int_{\mathbb{R}^2}\mathcal{X}_{\{\theta_c(t,x)>0\}}\,dx\right). \tag{2.3}$$

With the energy inequality in hand, we obtain the desired nonlinear recurrence relation on the superlevel sets of energy. Let $L > 1$ be specificed later, and put $L_k = L(1-2^{-k})$, $\theta_k = (\theta - L_k)_+$ and $T_k = -1-2^{-k}$. Define

$$E_k = \sup_{t\in[T_k,0]}\int_{\mathbb{R}^2}\theta_k^2(t,x)\,dx + \int_{T_k}^0\int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}\theta_k(\tau,x)|^2\,dx\,d\tau.$$

Choose $s \in [T_{k-1}, T_k]$ and $t \in [T_k, 0]$. Integrating (2.3) from s to t yields

$$\begin{aligned}&\int_{\mathbb{R}^2}\theta_k^2(t,x)\,dx + \int_s^t\int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}\theta_k(\tau,x)|^2\,dx\,d\tau\\ &\quad\le \int_{\mathbb{R}^2}\theta_k^2(s,x)\,dx + C(M)\left(\int_s^t\int_{\mathbb{R}^2}|\theta_k(\tau,x)|\,dx\,d\tau + \int_s^t\int_{\mathbb{R}^2}\mathcal{X}_{\{\theta_k(\tau,x)>0\}}\,dx\,d\tau\right).\end{aligned}$$

Now taking the supremum on the left hand side, discarding the energy at time s, and averaging over $s \in [T_{k-1}, T_k]$ on the right hand side, we have

$$E_k \le C(M)2^k\left(\int_{T_{k-1}}^0\int_{\mathbb{R}^2}\theta_k(\tau,x)\,dx\,d\tau + \int_{T_{k-1}}^0\int_{\mathbb{R}^2}\mathcal{X}_{\{\theta_k(\tau,x)>0\}}\,dx\,d\tau\right). \tag{2.4}$$

We must control the right-hand side of (2.4) by E_{k-1} in a nonlinear fashion. First, note that Sobolev embedding gives that $H^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, and using this estimate to interpolate, we obtain

$$||\theta_k||_{L^3([T_k,0]\times\mathbb{R}^2)} \le E_k^{\frac{1}{2}}.$$

Next, we have that if $\theta_k > 0$, then $\theta_{k-1} \ge 2^{-k}L$ and $\mathcal{X}_{\{\theta_k>0\}} \le \frac{2^k}{L}\theta_{k-1}$. Therefore

$$\int_{T_{k-1}}^0\int_{\mathbb{R}^2}\theta_k\,dx\,d\tau \le \int_{T_{k-1}}^0\int_{\mathbb{R}^2}\theta_k\mathcal{X}^2_{\{\theta_k>0\}}\,dx\,d\tau \le \frac{4^k}{L^2}\int_{T_{k-1}}^0\int_{\mathbb{R}^2}\theta_{k-1}^3\,dx\,d\tau \le \frac{4^k}{L^2}E_{k-1}^{\frac{3}{2}}$$

and

$$\int_{T_{k-1}}^0\int_{\mathbb{R}^2}\mathcal{X}_{\{\theta_k>0\}}\,dx\,d\tau = \int_{T_{k-1}}^0\int_{\mathbb{R}^2}\mathcal{X}^3_{\{\theta_k>0\}}\,dx\,d\tau \le \frac{8^k}{L^3}\int_{T_{k-1}}^0\int_{\mathbb{R}^2}\theta_{k-1}^3\,dx\,d\tau \le \frac{8^k}{L^3}E_{k-1}^{\frac{3}{2}}.$$

Combining these estimates, we have

$$E_k \leq \frac{C(M)^k}{L^2} E_{k-1}^{\frac{3}{2}}.$$

Depending only on M, we can choose L to be large enough and use (2.4) to show that E_1 can be made small enough such that $\lim_{k\to\infty} E_k = 0$. Thus $||\theta_k||_{L^3([T_k,0]\times\mathbb{R}^2)}$ converges to zero as $k \to \infty$, and applying the Lebesgue dominated convergence theorem shows that $\theta \leq L$ on $[-1,0]\times\mathbb{R}^2$, proving the claim. $\square$

To accommodate the second De Giorgi lemma, we must reformulate the L^∞ bound. The nonlocality of the equation makes the zooming arguments more delicate; since the decrease in oscillation required for Hölder regularity will be nonlocal in nature, we cannot use a sharp cutoff as in Lemma 2.3.3. To address this, we will make use of a suitable cutoff function. Let $c(x) \in C^\infty(\mathbb{R}^2)$ such that $c \geq 0$, $c = 0$ on $B_{\frac{7}{4}}(0)$, $c(x) = (|x|^{\frac{1}{4}} - 2)_+$ for $|x| \geq 3$, and $c(x) \geq (|x|^{\frac{1}{4}} - 2)_+$ for $|x| \geq 2$. We claim that $||(-\overline{\Delta})^{\frac{1}{2}} c||_{L^\infty} < \infty$. Using that $\nabla c \in C^\alpha(\mathbb{R}^2)$ for any $\alpha < 1$, applying the Riesz transform, and using part (1) of Proposition A.0.3 shows that $(-\overline{\Delta})^{\frac{1}{2}} c \in BMO \cap \mathring{C}^\alpha(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$. This cutoff function introduces an additional difficulty in that the drift term does not disappear after multiplying the equation by $\theta - c(x)$ and integrating. Since $\partial_\nu \Psi_1$ is now L^∞ and $\overline{\nabla}^\perp \Psi_2 \in L^\infty$ by Proposition 2.2.2, the Riesz transform gives that $\overline{\nabla}^\perp \Psi \in$ BMO. Performing a change of variables which follows the mean value of the drift through time, the new drift term will be exponentially integrable. Since BMO bounds are invariant under rescalings as well, following the flow at each successive dilation provides the needed uniform estimates. With this in mind, we can obtain the following sharper L^∞ bound.

Lemma 2.3.4 (Local L^∞ bound)**.** *For any $C^* > 0$, there exists $\delta > 0$ such that the following holds. Let $\theta \in L^\infty([-2,0]; H^{\frac{5}{2}}(\mathbb{R}^2))$ be a solution to*

$$\partial_t \theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f$$

such that $\theta(t,x) \leq 1 + c(x)$ on $[-2,0]\times\mathbb{R}^2$, $\operatorname{div} u = 0$ and

$$||(-\overline{\Delta})^{-\frac{1}{4}} f||_{L^\infty([-2,0];C^{\frac{1}{2}})(\mathbb{R}^2)} + ||u||_{L^\infty([-2,0];L^4(B_2(0)))} \leq C^*.$$

If

$$|\{\theta > 0\} \cap ([-1,0] \times B_1(0))| < \delta,$$

then $\theta(t,x) \leq \frac{1}{2}$ *for* $(t,x) \in [-\frac{1}{2},0] \times B_{\frac{1}{2}}(0)$.

Proof. Let γ_k be a bump function compactly supported in $B_{\frac{1}{2}+2^{-k-1}}$, equal to $\frac{1}{2}+2^{-k-1}$ on $B_{\frac{1}{2}+2^{-k-2}}$, with $0 \leq \gamma_k \leq \frac{1}{2}+2^{-k-1}$ for all x, and $\gamma_k < \gamma_l$ for $k > l$. We will also impose that $|\overline{\nabla}\gamma_k| \leq C2^k$ and $|(-\overline{\Delta})^{\frac{1}{2}}\gamma_k| \leq Ck2^k$ (we provide a short justification of this condition in the appendix after the discussion of Proposition A.0.7). Define $\theta_k := (\theta - (1 + c - \gamma_k))_+$. We multiply the equation by θ_k and argue as before. First we record the following estimates:

$$\begin{aligned}\int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{2}}\theta\theta_k\,dx &\leq \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}}\theta_k|^2\,dx + \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{2}}(1+c-\gamma_k)\theta_k\,dx \\ &\leq \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}}\theta_k|^2\,dx + \left(||(-\overline{\Delta})^{\frac{1}{2}}c||_{L^\infty} + Ck2^k\right)\int_{\mathbb{R}^2}\theta_k\,dx \qquad (2.5)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^2} \left(u\cdot\overline{\nabla}\theta\right)\theta_k\,dx &= \int_{\mathbb{R}^2}\left(u\cdot\overline{\nabla}\theta_k\right)\theta_k\,dx + \int_{\mathbb{R}^2}\left(u\cdot\overline{\nabla}(1+c-\gamma_k)\right)\theta_k\,dx \\ &= \int_{\mathbb{R}^2}\left(u\cdot\overline{\nabla}(1+c-\gamma_k)\right)\theta_k\,dx \\ &\leq C(C^*)\left(||\overline{\nabla}c||_{L^\infty} + ||\overline{\nabla}\gamma_k||_{L^\infty}\right)\left(\int_{\mathbb{R}^2}\theta_k^{\frac{4}{3}}\,dx\right)^{\frac{3}{4}}. \qquad (2.6)\end{aligned}$$

after applying Hölder's inequality with $q = \frac{4}{3}$ on θ and $\frac{q-1}{q} = 4$ on u. In addition, we can estimate the term

$$\int_{\mathbb{R}^2} f\theta_k\,dx \qquad (2.7)$$

using Proposition 2.3.2. Combining estimates for (2.5), (2.6), and (2.7), we arrive at the energy inequality

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}\theta_k^2\,dx + \int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}\theta_k|^2\,dx \leq \\ &\qquad C(C^*)\left(2^k + k2^k\right)\left(\int_{\mathbb{R}^2}\theta_k\,dx + \int_{\mathbb{R}^2}\mathcal{X}_{\{\theta_k>0\}}\,dx + \Big(\int_{\mathbb{R}^2}\theta_k^q\,dx\Big)^{\frac{1}{q}}\right). \qquad (2.8)\end{aligned}$$

Now define $T_k := -\frac{1}{2} - 2^{-k-1}$, and put

$$E_k := \sup_{t\in[T_k,0]} \int_{\mathbb{R}^2} \theta_k^2(t,x)\,dx + \int_{T_k}^0 \int_{\mathbb{R}^2} |(-\overline{\Delta})^{\frac{1}{4}}\theta_k(\tau,x)|^2\,dx\,d\tau.$$

Integrating in time, we have that the first two terms on the right hand side can be estimated as in Lemma 2.3.3. Using Jensen's inequality and the fact that $\mathcal{X}_{\{\theta_k>0\}} \le 2^k\theta_{k-1}$, we can estimate the third term on the right hand side of (2.8) by

$$\begin{aligned}
\int_{T_{k-1}}^0 \left(\int_{\mathbb{R}^2} \theta_k^q\,dx\right)^{\frac{1}{q}} d\tau &\le \int_{T_{k-1}}^0 \left(\int_{\mathbb{R}^2} \theta_k^q \mathcal{X}_{\{\theta_k>0\}}^{3-q}\,dx\right)^{\frac{1}{q}} d\tau \\
&\le C\left(\int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_k^q \mathcal{X}_{\{\theta_k>0\}}^{3-q}\,dx\,d\tau\right)^{\frac{1}{q}} \\
&\le C\left(\left(2^k\right)^{3-q} \int_{T_{k-1}}^0 \int_{\mathbb{R}^2} \theta_{k-1}^3\,dx\,d\tau\right)^{\frac{1}{q}} \\
&\le C^k E_{k-1}^{\frac{3}{2q}}.
\end{aligned}$$

Using the integrability assumption on u, and recalling that $q = \frac{4}{3}$, the nonlinear recurrence relation on E_k follows as in Lemma 2.3.3. Noticing that (2.8) shows that choosing δ arbitrarily small makes E_0 arbitrarily small, there exists δ such that $\lim_{k\to\infty} E_k = 0$. Therefore, θ_k converges to 0 in L^2 for every time $t \in [-\frac{1}{2}, 0]$. Thus $\theta \le \frac{1}{2}$ on $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}(0)$. $\square$

2.3.2 From L^∞ to C^α

With the L^∞ bound in hand, we turn to the second half of the De Giorgi argument. Let us remark the argument in [16] works for kernels comparable to the fractional Laplacian $(-\overline{\Delta})^\alpha$ raised to any power $\alpha \in (0,1)$. In addition, Schwab and Silvestre [86]) proved a regularity result for parabolic equations assuming that the drift and the forcing were bounded. Without bounded drift and forcing, we cannot follow [86]. However, we have more dissipation than is necessary for the argument in [16]. Therefore, we can instead make a compactness argument following [96]. Since $\alpha = \frac{1}{2}$, the solutions belong to $H^{\frac{1}{2}}$, and we can make use of Proposition A.0.6. This will show that the energy cannot increase or decrease too rapidly in time.

First, a parabolic version of the isoperimetric lemma will be shown, following the proof in [96]. This will then imply that θ enjoys a geometric rate of decrease in oscillation. Let ϕ be a compactly supported, radially symmetric and decreasing, C^∞ bump function such that $0 \leq \phi(x) \leq 1$ for all x, $\phi = 1$ on $B_1(0)$, and $\operatorname{supp}\phi \subset B_{\frac{3}{2}}(0)$. Let $\phi_0(x) = 1+c(x)-\phi(x)$, and $\phi_1(x) = 1 + c(x) - \frac{1}{2}\phi(x)$.

Lemma 2.3.5 (Isoperimetric Lemma)**.** *For any $C^*, \beta > 0$ there exists α such that the following holds. Let $\theta \in L^\infty([-2,0]; H^{\frac{5}{2}}(\mathbb{R}^2))$ solve*

$$\partial_t\theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f$$

with $\theta(x) \leq 1 + c(x)$. Assume that

$$||(-\overline{\Delta})^{-\frac{1}{4}} f||_{L^\infty([-2,0];C^{\frac{1}{2}})(\mathbb{R}^2)} + ||u||_{L^\infty([-2,0];L^4(B_2(0)))} \leq C^*$$

and $\operatorname{div} u = 0$. Fix δ as in Lemma 2.3.4. Using ϕ_0 and ϕ_1 as defined immediately above, let

$$\begin{aligned} A =&\{\theta > \frac{1}{2}\} \cap ([-1,0] \times B_1) \\ C =&\{\theta \leq 0\} \cap ([-2,-1] \times B_1) \\ D =&\{\phi_0 < \theta \leq \phi_1\} \cap ([-2,0] \times B_2). \end{aligned}$$

Then if $|A| \geq \delta$, $|C| \geq \beta$, then $|D| \geq \alpha$.

Proof. Assume that the lemma is false. Then, given β there exists a sequence of solutions θ_j such that $|A_j| \geq \delta$, $|C_j| \geq \beta$, $|D_j| \leq \frac{1}{j}$ with A_j, C_j, and D_j defined analogously to A, C, and D. Put $v_j = (\theta_j - \phi_0)_+$. The proof will use the Aubin-Lions compactness lemma [2] to extract a subsequential limit which will satisfy the energy inequality but does not take values in between 0 and $\frac{1}{2}$, reaching a contradiction.

In order to apply the Aubin-Lions lemma to v_j^2, we show that $\partial_t v_j^2 \in L^1([-2,0]; H^{-2}(B_2(0)))$, $v_j^2 \in L^2(H^{\frac{1}{2}}(B_2(0)))$, and $v_j^2 \in L^\infty([-2,0]; L^2(B_2(0)))$. The third criterion is immediate from the assumptions, so we focus on the first and second. We multiply by v_j and integrate. The

$L^\infty(L^4)$ bound on u gives that

$$\begin{aligned}\int_{\mathbb{R}^2}(u\cdot\overline{\nabla}\theta_j)v_j\,dx &= \int_{\mathbb{R}^2}(u\cdot\overline{\nabla}v_j)v_j\,dx + \int_{\mathbb{R}^2}\big(u\cdot\overline{\nabla}(1+c-\phi)\big)\,v_j\,dx\\ &= \int_{\mathbb{R}^2}\big(u\cdot\overline{\nabla}(1+c-\phi)\big)\,v_j\,dx\\ &\le C(C^*,\phi,c)\end{aligned}$$

after using the compact support of v_j, the bounds on c and ϕ, and Hölder's inequality. We can estimate the forcing term using Proposition 2.3.2 by setting $f=g$ and $v_j=\omega$. Using the L^∞ bound on v_j and absorbing the $\mathring{H}^{\frac{1}{2}}$ norm into the left hand side, we obtain the energy inequality

$$\frac{d}{dt}\int_{\mathbb{R}^2}v_j^2+\int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}v_j|^2\le C(C^*,\phi,c).$$

Integrating from s to t in time for $-2<s<t<0$ gives

$$\int_{\mathbb{R}^2}v_j^2(t)+\int_s^t\int_{\mathbb{R}^2}|(-\overline{\Delta})^{\frac{1}{4}}v_j|^2\le\int_{\mathbb{R}^2}v_j^2(s)+C(C^*,\phi,c)(t-s). \tag{2.9}$$

This implies that v_j is uniformly bounded in $L^2([-2,0];\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2))$. Also, note that since $0\le v_j\le 1$, for all x,y we have

$$|v_j^2(x)-v_j^2(y)|^2\le 4|v_j(x)-v_j(y)|^2$$

Examining the Gagliardo seminorm shows then that $||v_j^2||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}\le 4||v_j||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}$ (see [43] for details concerning equivalent definitions of fractional Sobolev spaces). By restriction, we have that $||v_j^2||_{\mathring{H}^{\frac{1}{2}}(B_2(0))}\le 4||v_j||_{\mathring{H}^{\frac{1}{2}}(B_2(0))}$, and so v_j^2 is uniformly bounded in $L^2([-2,0];H^{\frac{1}{2}}(B_2(0)))$ by a constant depending only on C^*,ϕ,c.

We now show that $\partial_t v_j^2\in L^1([-2,0];H^{-2}(B^2(0)))$; here $H^{-2}(B_2(0))$ denotes the dual of $H_0^2(B_2(0))$. Multiplying the equation by v_j, we obtain

$$\frac{1}{2}\partial_t v_j^2=-\operatorname{div}(u\theta_j)v_j-(-\overline{\Delta})^{\frac{1}{2}}\theta_j v_j+fv_j.$$

We must show that each term on the right hand side belongs to $L^1([-2,0];H^{-2}(B_2(0)))$.

1. Note that $\operatorname{div}(u\theta_j)v_j=\frac{1}{2}\operatorname{div}(uv_j^2)+u\cdot\nabla\phi_0 v_j$. Since $v_j^2\in L^\infty([-2,0]\times\mathbb{R}^2)$ is compactly supported and $u\in L^\infty([-2,0];L^4(\mathbb{R}^2))$, part (2) of Proposition A.0.3 shows that

$\mathrm{div}(uv_j^2)$ belongs to $L^1([-2,0];\mathring{H}^{-1}(\mathbb{R}^2) \subset L^1([-2,0];H^{-2}(B_2(0)))$. Also, since $\nabla\psi_0$ is smooth and compactly supported, it is immediate that $u\cdot\nabla\phi_0 v_j \in L^1([-2,0];L^2(\mathbb{R}^2))$. Therefore $\mathrm{div}(u\theta_j)v_j \in L^1([-2,0];H^{-2}(B_2(0)))$.

2. Since $(-\overline{\Delta})^{-\frac{1}{4}}f \in L^\infty([-2,0];L^\infty(\mathbb{R}^2))$ and $v_j \in L^\infty([-2,0]\times\mathbb{R}^2)\cap L^\infty([-2,0];\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2))$ is compactly supported in $B_2(0)$, we can apply part (1) of Proposition A.0.5 with $z = v_j$ and $f = w$ to conclude that fv_j is uniformly bounded in $L^1([-2,0];H^{-2}(\mathbb{R}^2)) \subset L^1([-2,0];H^{-2}(B_2(0)))$.

3. We have that

$$(-\overline{\Delta})^{\frac{1}{2}}\theta_j v_j = (-\overline{\Delta})^{\frac{1}{2}}[(\theta_j-\phi_0)-(\theta_j-\phi_0)_+]v_j + (-\overline{\Delta})^{\frac{1}{2}}v_j v_j + (-\overline{\Delta})^{\frac{1}{2}}\phi_0 v_j.$$

Since $(-\overline{\Delta})^{\frac{1}{2}}\phi_0 \in L^\infty([-2,0];L^2(\mathbb{R}^2))$ and $v_j \in L^\infty([-2,0];L^2(\mathbb{R}^2))$, it is immediate that $(-\overline{\Delta})^{\frac{1}{2}}\phi_0 v_j \in L^1([-2,0];L^2(B_2(0)))$. In addition, we can apply part (2) of Proposition A.0.5 to the second term to obtain that $(-\overline{\Delta})^{\frac{1}{2}}v_j v_j \in L^1([-2,0];H^{-2}(B_2(0))$. In order to estimate the first term, first note that the pointwise estimate of Córdoba and Córdoba [38] shows that $(-\overline{\Delta})^{\frac{1}{2}}[(\theta_j-\phi_0)-(\theta_j-\phi_0)_+]v_j$ is a positive measure on $B_3(0)$ for each time $t\in[-2,0]$. We first show that $(-\overline{\Delta})^{\frac{1}{2}}[(\theta-\phi_0)-(\theta-\phi_0)_+]v_j \in L^1([-2,0];\mathcal{M}(B_3(0)))$; here $\mathcal{M}(B_3(0))$ is the Banach space of all Borel measures on $B_3(0)$ with the total variation norm. We have that

$$(-\overline{\Delta})^{\frac{1}{2}}[(\theta-\phi_0)-(\theta-\phi_0)_+]v_j = -\frac{1}{2}\partial_t v_j^2 - (u\cdot\overline{\nabla}\theta_j)v_j - (-\overline{\Delta})^{\frac{1}{2}}v_j v_j - (-\overline{\Delta})^{\frac{1}{2}}\phi_0 v_j + fv_j.$$

To show that $(-\overline{\Delta})^{\frac{1}{2}}[(\theta-\phi_0)-(\theta-\phi_0)_+]v_j \in L^1([-2,0];\mathcal{M}(B_3(0)))$, we multiply by $\mathcal{X}_{B_3(0)}$ and integrate in space and time. Note that since each term on the right hand side contains a factor of v_j which is compactly supported in $B_2(0)$, multiplying by $\mathcal{X}_{B_3(0)}$ has no effect. First, we have that

$$\int_{-2}^0\int_{B_3(0)}\partial_t v_j^2 = \int_{B_3(0)}\int_{-2}^0\partial_t v_j^2 = \int_{B_3(0)} v_j^2(0) - v_j^2(-2) \le 2|B_3(0)|.$$

Here we have used the a priori regularity assumptions, the $\mathring{H}^{\frac{1}{2}}$ bound on v_j, and the equality

$$\frac{1}{2}\partial_t v_j^2 = -u\cdot\overline{\nabla}\theta_j v_j - (-\overline{\Delta})^{\frac{1}{2}}\theta_j v_j + fv_j$$

to justify integrating $\partial_t v_j^2$ in space and time. Next, splitting $u \cdot \overline{\nabla}\theta_j v_j = u \cdot \overline{\nabla}\phi_0 v_j + u \cdot \overline{\nabla} v_j v_j$ and integrating by parts shows that

$$\int_{-2}^{0}\int_{B_3(0)} u \cdot \overline{\nabla}\theta_j v_j \leq ||u||_{L^\infty(L^4)}.$$

Since $(-\overline{\Delta})^{\frac{1}{2}}\phi_0 v_j$ is bounded, multiplying by $\mathcal{X}_{B_3(0)}$ and integrating produces at most a fixed constant depending only on ϕ_0. Also, we have that

$$-\int_{-2}^{0}\int_{B_3(0)} (-\overline{\Delta})^{\frac{1}{2}} v_j v_j$$

is negative and may be discarded. Finally, applying Proposition 2.3.2 with $g = f$, $\omega = v_j$ and using the $L^2(\mathring{H}^{\frac{1}{2}})$ and L^∞ bounds on v_j shows that

$$\int_{-2}^{0}\int_{B_3(0)} f v_j \leq C(C^*, \phi, c).$$

Therefore, $(-\overline{\Delta})^{\frac{1}{2}}[(\theta - \phi_0) - (\theta - \phi_0)_+]v_j$ is bounded in $L^1([-2,0]; \mathcal{M}(B_3(0)))$ by a constant depending only on C^*, ϕ, c. By Sobolev embedding, $H_0^2(B_2(0)) \subset C_c(B_3(0))$. Recalling then that $\mathcal{M}(B_3(0)) = C_c(B_3(0))^*$, we have that $L^1([-2,0]; \mathcal{M}(B_3(0))) \subset L^1([-2,0]; H^{-2}(B_2(0)))$, showing that $(-\overline{\Delta})^{\frac{1}{2}}[(\theta - \phi_0) - (\theta - \phi_0)_+]v_j$, and therefore $(-\overline{\Delta})^{\frac{1}{2}}\theta_j v_j$, belong to $L^1([-2,0]; H^{-2}(B_2(0)))$.

Since $H^{\frac{1}{2}}(B_2(0))$ embeds compactly into $L^1(B_2(0))$ (see again [43]) and $L^1(B_2(0))$ embeds continuously into $H^{-2}(B_2(0))$, by the Aubin-Lions compactness lemma from [2], up to a subsequence, v_j^2 converges in $L^2([-2,0]; L^1(B_2(0)))$ to a function v^2. Passing to the limit in (2.9) shows that v^2 satisfies the inequality

$$\int_{B_2} v^2(t) \leq \int_{B_2} v^2(s) + C(C^*, \phi, c)(t-s)$$

for $s < t$. By assumption, θ_j satisfies $|D_j| \leq \frac{1}{j}$. Therefore, v_j then satisfies by definition

$$|\{0 < v_j \leq \frac{1}{2}\phi\} \cap ([-2,0] \times B_2)| \leq \frac{1}{j},$$

and v_j^2 satisfies

$$|\{0 < v_j^2 \leq \frac{1}{4}\phi^2\} \cap ([-2,0] \times B_2)| \leq \frac{1}{j}.$$

Using Tchebyshev's inequality and passing to the limit, we have that

$$|\{0 < v^2 < \frac{1}{4}\phi^2\} \cap ([-2,0] \times B_2)| = 0.$$

Since v^2 belongs to $L^2([-2,0]; H^{\frac{1}{2}}(\operatorname{supp}\phi))$, applying Proposition A.0.6 to v^2 shows that for almost every time $t \in [-2,0]$, either $v^2 = 0$ or $v^2 \geq \frac{1}{4}\phi^2$. Using that $|C_j| \geq \beta$ for every j, there must exist a positive measure set of times in $[-2,-1]$ for which $v = 0$. The energy inequality shows that as soon as $v = 0$ for some time s, $v = 0$ on all of $[s,0] \times B_2$, and thus we conclude that $v = 0$ on $[-1,0] \times B_2$. However, we also have that $|A_j| \geq \delta$ for all j, which bounds the norms of v_j^2 in $L^2([-1,0]; L^1(B_2(0)))$ from below uniformly in j, contradicting the convergence of v_j^2 to v^2 in $L^2([-2,0]; L^1(B_2(0)))$. □

We turn now to the oscillation lemma.

Lemma 2.3.6 (Decrease in Oscillation)**.** *For any C^*, there exists $\zeta \in (0,1)$, $\epsilon \in (0,\frac{1}{4})$, and $\eta > 0$ such that the following holds. Let $\theta \in L^\infty([-2,0]; H^{\frac{5}{2}}(\mathbb{R}^2))$ solve*

$$\partial_t \theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f$$

with $\theta(x) \leq 1 + c_\epsilon(x) := 1 + (|x|^\epsilon - 2^{4\epsilon})_+$. Assume that

$$\eta^{-1}||(-\overline{\Delta})^{-\frac{1}{4}} f||_{L^\infty([-2,0];C^{\frac{1}{2}}(\mathbb{R}^2))} + ||u||_{L^\infty([-2,0];L^4(B_2(0)))} \leq C^*$$

and $\operatorname{div} u = 0$. Let δ, β, α be as in Lemma 2.3.5 with $|C| \geq \beta$. Then $\theta \leq 1 - \zeta$ on $[-\frac{1}{2},0] \times B_{\frac{1}{2}}$.

Proof. Choose K such that $K\alpha > |(-2,0) \times B_2|$, and let $\eta = 2^{-K}$. Put $\theta_k = 2^k(\theta - (1-2^{-k}))$. By scaling, θ_k solves the equation

$$\partial_t \theta_k + u \cdot \overline{\nabla}\theta_k + (-\overline{\Delta})^{\frac{1}{2}}\theta_k = 2^k f.$$

For $k \leq K$,

$$||2^k(-\overline{\Delta})^{-\frac{1}{4}} f||_{L^\infty(C^{\frac{1}{2}})} \leq C^*.$$

Choose $\epsilon << \frac{1}{4}$ to be small enough such that

$$2^K(|x|^\epsilon - 2^{4\epsilon})_+ \leq (|x|^{\frac{1}{4}} - 2)_+ \leq c(x)$$

for all x. Note that since $k \le K$ we have

$$\theta_k(x) \le 1 + 2^K c_\epsilon(x) \le 1 + c(x).$$

Fix $k \le K$ now, and suppose that

$$|\{\theta_{j+1} > 0\} \cap ([-1, 0] \times B_1)| \ge \delta. \tag{2.10}$$

for all $j \le k$. This implies that

$$|\{\theta_j > \frac{1}{2}\} \cap ([-1, 0] \times B_1)| \ge \delta.$$

Since $|\{\theta_j \le 0\} \cap \{[-2, -1] \times B_1\}| \ge \beta$ for all j, we have that by Lemma 2.3.5,

$$|\{\phi_0 < \theta_j \le \phi_1\} \cap ([-2, 0] \times B_2)| \ge \alpha.$$

Noticing that the sets $\{\phi_0 < \theta_j \le \phi_1\}$, $\{\phi_0 < \theta_{j'} \le \phi_1\}$ are disjoint for $j \ne j'$, we have that (2.10) cannot hold for $k = K$ by choice of K. So there must exist $k < K$ for which

$$|\{\theta_{k+1} > 0\} \cap ([-1, 0] \times B_1)| < \delta.$$

Then by Lemma 2.3.4, $\theta_{k+1} \le \frac{1}{2}$ in $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$, and $\theta \le 1 - 2^{-(2+K)}$ in $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$, proving the claim with $\zeta = 2^{-(2+K)}$. $\square$

We have arrived at Lemma 2.3.7 as an easy corollary.

Lemma 2.3.7. *If $-1 - c_\epsilon \le \theta \le 1 + c_\epsilon$ on $[-2, 0] \times \mathbb{R}^2$ and the conditions of Lemma 2.3.6 are satisfied, then on $[-\frac{1}{2}, 0] \times B_{\frac{1}{2}}$,*

$$\sup \theta - \inf \theta \le 2 - \zeta.$$

We can now prove the main regularity estimate for $\partial_\nu \Psi$.

Proof of Lemma 2.3.1. Throughout the argument, θ, u, and f correspond to (dilated versions of) $\partial_\nu \Psi$, $\overline{\nabla}^\perp \Psi$, and $\overline{\Delta} \Psi_2$, respectively. The regularity assumptions on each of the function in the De Giorgi lemmas is provided by Proposition 2.2.2; we give the details in the proof as they appear. Recall that the local existence theorem guarantees the existence of some time $\bar{T}$ such that (QG) admits a smooth solution on $[0, \bar{T}]$. We assume that $\nabla\Psi \in L^\infty([0,T]; H^{\frac{5}{2}}(\mathbb{R}^2))$ is a solution to (QG), and $T \geq \bar{T}$. We then choose $(t_0, x_0) \in [0,T] \times \mathbb{R}^2$ such that $t_0 \geq \frac{\bar{T}}{2}$. We define $K_0 = \inf(1, \frac{t_0}{4})$. Put $\theta_0(t,x) = \partial_\nu \Psi(t_0 + K_0 t, x_0 + K_0 x)$, $u_0(t,x) = \nabla^\perp \Psi(t_0 + K_0 t, x_0 + K_0 x)$, and $f_0(t,x) = \overline{\Delta}\Psi_2(t_0 + K_0 t, x_0 + K_0 x)$. Then by the a priori estimates in (5) and (6) of Proposition 2.2.2, θ_0 (and $-\theta_0$) satisfy the assumptions of Lemma 2.3.3, and we have that $\theta_0 \in L^\infty([-1,0] \times \mathbb{R}^2)$. Since the argument is translation invariant in space and we can only need to consider times $t \in [\frac{\bar{T}}{2}, T]$, we have in fact that $\theta \in L^\infty([0,T] \times \mathbb{R}^2)$, with $||\theta||_{L^\infty([0,T]\times\mathbb{R}^2)}$ depending only on $||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$.

Continuing to fix (t_0, x_0) and K_0 as above, we will show that θ is Hölder continuous at (t_0, x_0). We will inductively define a sequence of dilated functions for some factor of dilation K to be determined later. Let $\Gamma_1(t)$ be the solution to the ODE

$$\begin{cases} \dot{\Gamma}_1(t) = \fint_{B_1(\Gamma_1(t))} u_0(Kt, Ky)\, dy \\ \Gamma_1(0) = 0 \end{cases}$$

and put

$$\theta_1(t,x) = \frac{\theta_0(Kt, Kx + \Gamma_1(t))}{||\theta||_{L^\infty}}$$

$$u_1(t,x) = u_0(Kt, Kx + \Gamma_1(t))$$

$$\overline{\Delta}\Psi_{2,1}(t,x) = \frac{1}{||\theta||_{L^\infty}} f_0(Kt, Kx + \Gamma_1(t)).$$

For $k > 1$, define

$$\begin{cases} \dot{\Gamma}_{k+1}(t) = \fint_{B_1(\Gamma_{k+1}(t))} u_k(Kt, Ky)\, dy - \fint_{B_1(0)} u_k(t) \\ \Gamma_{k+1}(0) = 0 \end{cases}$$

$$\theta_{k+1}(t,x) = \frac{1}{1 - \frac{\varsigma}{4}} \left(\theta_k \left(Kt, Kx + \Gamma_{k+1}(t)\right) - \frac{1}{2} \left(\sup_{[-\frac{1}{4},0]\times B_{\frac{1}{4}}} \theta_k + \inf_{[-\frac{1}{4},0]\times B_{\frac{1}{4}}} \theta_k \right) \right)$$

$$u_{k+1}(t,x) = u_k \left(Kt, Kx + \Gamma_{k+1}(t)\right)$$

$$\overline{\Delta}\Psi_{2,k+1}(t,x) = \overline{\Delta}\Psi_{2,k}(Kt, Kx + \Gamma_{k+1}(t)).$$

We have that θ_k solves the equation

$$\partial_t \theta_k + (u_k - \int_{B_1(0)} u_k) \cdot \overline{\nabla}\theta_k + (-\overline{\Delta})^{\frac{1}{2}}\theta_k = \left(\frac{K}{(1 - \frac{\varsigma}{4})}\right)^k \overline{\Delta}\Psi_{2,k}.$$

Examining the assumptions of the De Giorgi lemmas (Lemma 2.3.4, Lemma 2.3.5, Lemma 2.3.6, and Lemma 2.3.7), we see that K is subject to the following constraints.

1. K needs to be small enough to satisfy
$$\frac{1}{1-\frac{\zeta}{2}}c_\epsilon(Kx) < c_\epsilon(x)$$
for all $x \geq \frac{1}{K}$. Recalling that $c_\epsilon(x) = (|x|^\epsilon - 2^{4\epsilon})_+$, choosing $K \leq 1 - \frac{\zeta}{2}$ satisfies this constraint.

2. K should be small enough so that $\left(\frac{K}{(1-\frac{\zeta}{4})}\right)^k \overline{\Delta}\Psi_{2,k}$ satisfies the assumptions of the De Giorgi lemmas uniformly in k. Specifically, we need $(-\overline{\Delta})^{-\frac{1}{4}}\left(\left(\frac{K}{(1-\frac{\zeta}{4})}\right)^k \overline{\Delta}\Psi_{2,k}\right) \in L^\infty([-2,0];C^{\frac{1}{2}}(\mathbb{R}^2))$ to have small norm. Applying $(-\overline{\Delta})^{-\frac{1}{4}}$ divides by a factor of $K^{\frac{k}{2}}$, but we can choose K to be very small compared to $(1-\frac{\zeta}{4})$, and by (3) and (5) of Proposition 2.2.2, we can choose K to satisfy this constraint.

3. We must ensure that $u_k - \fint_{B_1(0)} u_k$ satisfies the assumptions of the De Giorgi lemmas uniformly in k. Specifically, we must have that $u_k - \fint_{B_1(0)} u_k \in L^\infty([-2,0];L^4(B_2(0)))$ uniformly in k. Using that $\overline{\nabla}^\perp\Psi_1$ is related to θ by the Riesz transform, the L^∞ bound on θ, part (7) of Proposition 2.2.2, parts (1) and (2) of Proposition A.0.9, and the scale invariance of the BMO norm, this condition is satisfied independent of K.

4. Notice that each successive dilation includes a change of variables which follows the new flow of the dilated drift term. At the k^{th} iteration, we will obtain a decrease in oscillation for θ_k on the set $[-\frac{1}{2},0]\times B_{\frac{1}{2}}(0)$. Then after dilating by K and shifting according to Γ_{k+1}, we must ensure that $-1-c_\epsilon \leq \theta_{k+1} \leq 1+c_\epsilon$. Applying Proposition A.0.9, we have that $|\dot{\Gamma}_{k+1}| < C$ for some fixed constant C. Therefore we can choose K small enough so that zooming in by a factor of K and then shifting according to the new drift gives that $-1-c_\epsilon \leq \theta_{k+1} \leq 1+c_\epsilon$.

We choose K to satisfy the above constraints. Thus we have that $\{\theta_k\}_{k=1}^\infty$ satisfies the assumptions of the De Giorgi lemmas uniformly in k, and we obtain a decrease in oscillation of $1-\frac{\zeta}{4}$ for each successive iteration. To see that θ is Hölder continuous, put

$$U_k = \sup_{[-2,0]} |\dot{\Gamma}_k(t)|$$

and notice that the set $[-\frac{1}{2},0] \times B_{\frac{1}{2}}(\Gamma_k(t))$ contains the rectangle $[-\frac{1}{4U_k},0] \times B_{\frac{1}{4}}(0)$. By Proposition A.0.9, there exists U such that $U_k \leq U$ for all k. Putting $D = \min\left(\frac{K}{4}, \frac{1}{8U}\right)$, we have that if (t,x) is such that

$$|(t,x)-(t_0,x_0)| \approx D^k$$

then

$$|\theta(t,x)-\theta(t_0,x_0)| \leq \left(1-\frac{\zeta}{4}\right)^k.$$

Therefore we have that θ is Hölder continuous at (t_0,x_0) with exponent

$$r = \frac{\log(1-\frac{\zeta}{4})}{\log(D)}.$$

We have that r depends only on the parameters M and C^*, which in turn depend only on $||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$. In addition, r does not depend on the choice of (t_0,x_0); in particular, θ is uniformly C^r throughout the interval $[0,T]\times\mathbb{R}^2$, so the lemma is complete. □

2.4 Bootstrapping from C^α to $\mathring{B}^1_{\infty,\infty}$

We now show that $\partial_\nu\Psi_1(t,x) \in L^\infty([0,T];\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$, which will give that $\overline{\nabla}^\perp\Psi_1 \in L^\infty([0,T]\times[0,\infty);\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. Here $[0,T]\times[0,\infty)$ denotes t and z, and $\mathbb{R}^2$ includes points $x=(x_1,x_2)$ belonging to flat planes $z=z_0$. Due to the fact that the Poisson kernel is the fundamental solution of the equation

$$\partial_t\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = 0 \tag{2.11}$$

we need the following two lemmas. These lemmas provide estimates on the regularity of the solution to an inhomogeneous version of (2.11). Let us remark that in the case of critical SQG, one can use either potential theory in the style of [17] or Littlewood-Paley arguments in the style of [29] to bootstrap the regularity. Also, both potential theory and Littlewood-Paley arguments can be used to show the sharp $\mathring{B}^1_{\infty,\infty}$ bound coming from the forcing term.

However, the most direct method in our situation seems to be to use potential theory for the nonlinear term and Littlewood-Paley arguments for the forcing.

Roughly speaking, the following lemma will show that the regularity of the nonlinear terms is additive; if θ is Hölder continuous in space-time with exponent α_1 and u in space with exponent α_2, then the convolution of their product with the Poisson kernel in space-time is Hölder continuous in space with exponent $\alpha_1 + \alpha_2$. Let us give an intuition as to why such a statement should hold. Given functions $f \in C^{\alpha_1}$, $g \in C^{\alpha_2}$, then $fg \in C^{\alpha_1 \wedge \alpha_2}$. However, if $f(x_0) = g(x_0) = 0$, then

$$|f(x)g(x) - f(x_0)g(x_0)| = |(f(x) - f(x_0))(g(x) - g(x_0))| \leq |x - x_0|^{\alpha},$$

and so fg is C^{α} at x_0. If we are trying to increase the regularity at (t_0, x_0), we can ensure that $\theta(t_0, x_0) = u(t, x_0) = 0$ for all $t \in [0, t_0]$ by performing a change of variables which follows the characteristics. Then the nonlinear term is effectively C^{α}, allowing us to bootstrap the regularity up to the space $\mathring{B}^1_{\infty,\infty}$.

Lemma 2.4.1. *Let $f(t, x) \in C^{\alpha_1}([0, t_0] \times \mathbb{R}^2)$, $h(t, x) \in L^{\infty}([0, t_0]; C^{\alpha_2}(\mathbb{R}^2))$, and $f(t_0, 0) = h(t, 0) = 0$ for all $t \in [0, t_0]$. Let $\mathcal{P}(t, x)$ be the Poisson kernel (extended to equal 0 for t negative). Let $\alpha = \alpha_1 + \alpha_2$, and define*

$$g(t, x) = \int_0^t \int_{\mathbb{R}^2} \mathcal{P}(t - s, x - y) \operatorname{div}(h(s, y) f(s, y)) \, dy \, ds.$$

1. *If $0 < \alpha < 1$, then*

$$\sup_{x \in \mathbb{R}^2} \frac{|g(t_0, x) - g(t_0, 0)|}{|x|^{\alpha}} < C||f||_{C^{\alpha_1}} ||h||_{L^{\infty}(C^{\alpha_2})}.$$

2. *If $1 \leq \alpha < 2$, then*

$$\sup_{x \in \mathbb{R}^2} \frac{|g(t_0, x) - 2g(t_0, 0) + g(t_0, -x)|}{|x|^{\alpha}} < C||f||_{C^{\alpha_1}} ||h||_{L^{\infty}(C^{\alpha_2})}.$$

Proof. Before starting, we remark that the gradient of the Poisson kernel

$$\nabla_x \mathcal{P}(t, x) = C \frac{tx}{(|x|^2 + t^2)^{\frac{5}{2}}}$$

is homogenous of degree 3 in $(-\infty, \infty) \times \mathbb{R}^2$. Using the fact that it is smooth away from the origin and has mean value zero in space over any set $\{t\} \times B(r, 0)$, we see that $\nabla_x \mathcal{P}$ is a singular integral in space-time. Beginning with the first case, we integrate by parts and split the integral around the singularity to obtain

$$
\begin{aligned}
g(t_0, 0) - g(t_0, x) &= \int_0^{t_0} \int_{\mathbb{R}^2} (\mathcal{P}(t_0 - s, -y) - \mathcal{P}(t_0 - s, x - y)) \operatorname{div}(h(s, y) f(s, y)) \, dy \, ds \\
&= -\int_0^{t_0} \int_{\mathbb{R}^2} (\nabla_x \mathcal{P}(t_0 - s, -y) - \nabla_x \mathcal{P}(t_0 - s, x - y)) h(s, y) f(s, y) \, dy \, ds \\
&= -\iint_{B_{3|x|}(t_0, 0)} \nabla_x \mathcal{P}(t_0 - s, -y) h(s, y) f(s, y) \, dy \, ds \\
&\quad + \iint_{B_{3|x|}(t_0, 0)} \nabla_x \mathcal{P}(t_0 - s, x - y) h(s, y) f(s, y) \, dy \, ds \\
&\quad - \iint_{(B_{3|x|}(t_0, 0))^{\complement}} (\nabla_x \mathcal{P}(t_0 - s, -y) - \nabla_x \mathcal{P}(t_0 - s, x - y)) h(s, y) f(s, y) \, dy \, ds \\
&= I + II + III
\end{aligned}
$$

We start with I; using the fact that $f(t_0, 0) = h(s, 0) = 0$, we integrate in polar coordinates in space-time to obtain

$$
\begin{aligned}
I &= -\iint_{B_{3|x|}(t_0, 0)} \nabla_x \mathcal{P}(t_0 - s, -y)(h(s, y) - h(s, 0))(f(s, y) - f(t_0, 0)) \, dy \, ds \\
&\leq C ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} \int_0^{3|x|} r^{\alpha_1 + \alpha_2 - 1} \, dr \\
&= C ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} |x|^\alpha.
\end{aligned}
$$

Moving to II, note that by the mean value condition on $\nabla_x \mathcal{P}$ and the assumptions on u and θ,

$$
\begin{aligned}
II = \iint_{B_{3|x|}(t_0, 0)} & \nabla_x \mathcal{P}(t_0 - s, x - y) h(s, y) f(s, y) \, dy \, ds \\
&= \iint_{B_{3|x|}(t_0, 0)} \nabla_x \mathcal{P}(t_0 - s, x - y) \left[(h(s, y) - h(s, 0))(f(s, y) - f(t_0, x)) \right] dy \, ds \\
&\quad + \iint_{B_{3|x|}(t_0, 0)} \nabla_x \mathcal{P}(t_0 - s, x - y) \left[(h(s, y) - h(s, x))(f(t_0, x) - f(t_0, 0)) \right] dy \, ds \\
&\leq ||h||_{L^\infty(C^{\alpha_2})} |x|^{\alpha_2} ||f||_{C^{\alpha_1}} \int_0^{3|x|} r^{\alpha_1 - 1} dr + ||f||_{C^{\alpha_1}} |x|^{\alpha_1} ||h||_{L^\infty(C^{\alpha_2})} \int_0^{3|x|} r^{\alpha_2 - 1} dr \\
&\leq C ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} |x|^\alpha.
\end{aligned}
$$

Finally, since the domain of integration for III is a fixed distance away from the singularity, a first order space-time Taylor estimate on $\nabla_x \mathcal{P}$ gives that on the domain of integration,

$$|\mathcal{P}(t_0 - s, x - y) - \mathcal{P}(t_0 - s, -y)| \leq \frac{|x|}{|(t_0 - s, x - y)|^4}.$$

Therefore, using the properties of u, θ, and $\nabla_x \mathcal{P}$ gives

$$\begin{aligned} III &\leq \iint_{(B_{3|x|}(t_0,0))^{\complement}} \big|(\nabla_x \mathcal{P}(t_0 - s, -y) - \nabla_x \mathcal{P}(t_0 - s, x - y))\cdot \\ &\qquad (h(s,y) - h(s,0)(f(s,y) - f(t_0,0)\big|\, dy\, ds \\ &\leq ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} \int_{3|x|}^{\infty} \frac{|x|^{\alpha+1}}{r^2}\, dr \\ &\leq C ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} |x|^\alpha \end{aligned}$$

Combining estimates for I, II, and III gives the result.

We now consider the case $1 \leq \alpha < 2$. As before, we integrate by parts and split the integral into two pieces;

$$\begin{aligned} g(t_0, x) - 2g(t_0, 0) + g(t_0, -x) &= \int_0^{t_0} \int_{\mathbb{R}^2} (\mathcal{P}(t_0 - s, x - y) - 2\mathcal{P}(t_0 - s, -y) \\ &\qquad + \mathcal{P}(t_0 - s, -x - y)) \cdot \operatorname{div}(h(s,y) f(s,y))\, dy\, ds \\ &= -\iint_{B(3|x|,0)} (\nabla_x \mathcal{P}(t_0 - s, x - y) - 2\nabla_x \mathcal{P}(t_0 - s, -y) \\ &\qquad + \nabla_x \mathcal{P}(t_0 - s, -x - y)) \cdot h(s,y) f(s,y)\, dy\, ds \\ &\quad - \iint_{(B(3|x|,0))^{\complement}} (\nabla_x \mathcal{P}(t_0 - s, x - y) - 2\nabla_x \mathcal{P}(t_0 - s, -y) \\ &\qquad + \nabla_x \mathcal{P}(t_0 - s, -x - y)) \cdot h(s,y) f(s,y)\, dy\, ds \\ &= I + II \end{aligned}$$

For the first piece, noticing that

$$g(t_0, x) - 2g(t_0, 0) + g(t_0, -x) = (g(t_0, x) - g(t_0, 0)) - (g(t_0, 0) - g(t_0, -x))$$

we can use the local estimate from the first part to conclude that $I \leq C ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} |x|^\alpha$.

For II, we can use the fact that

$$\nabla_x \mathcal{P}(t_0 - s, x - y) - 2\nabla_x \mathcal{P}(t_0 - s, -y) + \nabla_x \mathcal{P}(t_0 - s, -x - y)$$

vanishes to first order. Since the domain of integration in II avoids the singularity, a second order space-time Taylor expansion gives that in the domain of integration,

$$|\mathcal{P}(t_0 - s, x - y) - 2\mathcal{P}(t_0 - s, -y) + \mathcal{P}(t_0 - s, -x - y)| \leq \frac{|x|^2}{|(t_0 - s, -y)|^5}.$$

Therefore, we can write

$$\begin{aligned}
II \leq & \iint_{(B_{3|x|}(t_0,0))^{\complement}} \big|(\nabla_x \mathcal{P}(t_0 - s, x - y) - 2\nabla_x \mathcal{P}(t_0 - s, -y) + \nabla_x \mathcal{P}(t_0 - s, -x - y)) \cdot \\
& \qquad (h(s, y) - h(s, 0)(f(s, y) - f(t_0, 0)\big| \, dy \, ds \\
& \leq ||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} \int_{3|x|}^{\infty} \frac{|x|^{\alpha+2}}{r^3} \, dr \\
& \leq C||h||_{L^\infty(C^{\alpha_2})} ||f||_{C^{\alpha_1}} |x|^\alpha
\end{aligned}$$

concluding the proof of the second part.

□

We provide now a short proof of the estimate needed for the right hand side.

Lemma 2.4.2. *Let* $\omega \in L^\infty([0, T]; \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$*, and define for* $t \in [0, T]$

$$g(t, x) = \int_0^t \int_{\mathbb{R}^2} \mathcal{P}(t - s, x - y) \operatorname{div}(\omega(s, y)) \, dy \, ds.$$

Then $g \in L^\infty([0, T]; \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$*.*

Proof. We must show that $\sup_j 2^j ||\Delta_j g||_{L^\infty} < \infty$. Recall that Δ_j is a dilation in frequency by a factor of 2^j of a Fourier multiplier which isolates frequences on an annulus of radius 1. We let $\tilde{\Delta}_j$ be a dilation by a factor of 2^j of a Fourier multiplier which strictly contains the annulus of radius 1, ensuring that the frequency support of Δ_j is contained inside that of $\tilde{\Delta}_j$. Then we can write

$$\begin{aligned}
\Delta_j g(t, x) &= \int_0^t \int_{\mathbb{R}^2} \mathcal{P}(t - s, x - y) \operatorname{div}(\Delta_j \omega(s, y)) \, dy \, ds \\
&= \int_0^t \int_{\mathbb{R}^2} \nabla_x \tilde{\Delta}_j \mathcal{P}(t - s, x - y) \Delta_j \omega(s, y) \, dy \, ds \\
&\leq C \int_0^t 2^j e^{-(t-s)2^j} 2^{-j} ||\omega(s, \cdot)||_{\mathring{B}^1_{\infty,\infty}} \, ds \\
&\leq C 2^{-j} ||\omega||_{L^\infty(\mathring{B}^1_{\infty,\infty})}
\end{aligned}$$

$\square$

We can now show that the regularity of $\partial_\nu \Psi$ can be bootstrapped all the way to $\mathring{B}^1_{\infty,\infty}$. Let Ψ be a strong solution to (QG) on $[0,T]$. We have that $\partial_\nu \Psi = \theta$ satisfies

$$\partial_t \theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = -u \cdot \overline{\nabla}\theta + \overline{\Delta}\Psi_2.$$

From Lemma 2.3.1, we have that $\theta \in C^r([0,T]\times\mathbb{R}^2)$. From the Riesz transform, we have also that $\overline{\nabla}^\perp \Psi_1|_{z=0} \in L^\infty([0,T];C^r(\mathbb{R}^2))$. By interpolating (4) and (7) from Proposition 2.2.2, we have that for all $\alpha < 1$, $\nabla^\perp \Psi_2|_{z=0} \in L^\infty([0,T];C^\alpha(\mathbb{R}^2)) \cap L^\infty([0,T];\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. We can combine Lemma 2.4.1 with Lemma 2.4.2 to show that $\theta \in L^\infty(\mathring{B}^1_{\infty,\infty})$. In order to apply Lemma 2.4.1, we fix (t_0, x_0) and perform a change of variables which follows the flow. Specifically, let

$$\begin{cases} \dot{\Gamma}(t) = u(t, \Gamma(t)) \\ \Gamma(t_0) = x_0 \end{cases}$$

This trajectory is well-defined since we are on the interval for which (QG) has a smooth solution. Crucially, the argument relies only on the existence of $\Gamma(t)$ and the boundedness of $\dot{\Gamma}(t)$, not the smoothness. Define

$$\tilde{\theta}(t,x) = \theta(t, x+\Gamma(t)) - \theta(t_0, x_0)$$

$$\tilde{u}(x,t) = u(t, x+\Gamma(t))$$

$$\overline{\Delta}\tilde{\Psi}_2(t,x) = \overline{\Delta}\Psi_2(t, x+\Gamma(t)).$$

Then $\tilde{\theta}$ solves the equation

$$\partial_t \tilde{\theta}(t,x) + (-\overline{\Delta})^{\frac{1}{2}}\tilde{\theta}(t,x) = -(\tilde{u}(t,x) - \tilde{u}(t,0)) \cdot \overline{\nabla}\tilde{\theta}(t,x) + \overline{\Delta}\tilde{\Psi}_2(t,x).$$

The norms of $\tilde{\theta} \in C^r([0,T]\times\mathbb{R}^2)$, $\tilde{u} \in L^\infty([0,T];C^r(\mathbb{R}^2))$, and $\nabla^\perp \tilde{\Psi}_2|_{z=0} \in L^\infty([0,T];C^\alpha(\mathbb{R}^2)) \cap L^\infty([0,T];\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$ are preserved under this change of variables since $\dot{\Gamma}(t)$ is bounded. We split $\tilde{\theta} = g_0 + g_1 + g_2$, where

$$g_0(t,x) = \tilde{\theta}(0,\cdot) * \mathcal{P}_t(\cdot)(x)$$

$$\partial_t g_1 + (-\overline{\Delta})^{\frac{1}{2}} g_1 = -(\tilde{u} - \tilde{u}(t,0)) \cdot \overline{\nabla}\tilde{\theta}$$

$$\partial_t g_2 + (-\overline{\Delta})^{\frac{1}{2}} g_2 = \overline{\Delta}\tilde{\Psi}_2.$$

Since g_0 is a convolution with the Poisson kernel of a shifted version of $\tilde{\theta}$, its regularity depends only on that of the initial data. Focusing on the other two terms, we have that g_1 can be written using Duhamel's formula with $f(t,x) = \tilde{\theta}(t,x)$ and $h(t,x) = \tilde{u}(t,x) - \tilde{u}(t,0)$, satisfying the assumptions of Lemma 2.4.1. Therefore, g_1 is C^{2r} in space at (t_0, x_0). In addition, g_2 can also be written using Duhamel's formula with $\omega = \overline{\nabla}\tilde{\Psi}_2$, satisfying the assumptions of Lemma 2.4.2, and so $g_2 \in L^\infty(\mathring{B}^1_{\infty,\infty})$. Repeating the argument for arbitrary (t_0, x_0) and recalling that the difference quotient characterization of $\mathring{B}^1_{\infty,\infty}$ is locally stronger than C^{2r} for any $2r < 1$ shows that $\theta \in C^r([0,T] \times \mathbb{R}^2) \cap L^\infty([0,T]; C^{2r}(\mathbb{R}^2))$. Applying the Riesz transform combined with Proposition A.0.3 and Proposition A.0.4 shows that $\overline{\nabla}^\perp \Psi_1|_{z=0} \in L^\infty([0,T]; C^{2r}(\mathbb{R}^2))$. Recalling the a priori estimates in parts (4) and (7) of Proposition 2.2.2, we have also that $\nabla\Psi_2$, and therefore u, are in $L^\infty([0,T]; C^{2r}(\mathbb{R}^2))$. We then repeat the argument N times, for $Nr \geq 1$. On the last iteration, g_0 and g_1 become $C^{1,Nr-1}$; however, the regularity of g_2 becomes the limiting factor, since $g_2 \in L^\infty([0,T]; \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. We cannot bootstrap any higher, and thus we have shown that $\theta \in L^\infty([0,T]; \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$.

We now show that for any z, $\nabla\Psi_1(\cdot, z)$ enjoys the same regularity in x as $\partial_\nu \Psi_1$. Recalling that the $L^1(\mathbb{R}^2)$ norm of the Poisson kernel $\mathcal{P}_z(x) := \mathcal{P}(x,z)$ is equal to 1 for any z, we can say that for all j,

$$||\Delta_j \left(\mathcal{P}_z * (\partial_\nu \Psi_1)\right)||_{L^\infty(\mathbb{R}^2)} \leq ||\Delta_j \left(\partial_\nu \Psi_1\right)||_{L^\infty(\mathbb{R}^2)}$$

(where the Littlewood-Paley projection is in x only). This shows that $(\mathcal{P}_z * (\partial_\nu \Psi_1)) \in \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2)$ with norm less than or equal to that of $\partial_\nu \Psi_1$. Furthermore, this estimate is uniform in z. Next, we note that

$$\nabla\Psi_1(z,x) = \big(\mathcal{P}_z * (\partial_\nu \Psi_1)(x), \mathcal{R}_1(\mathcal{P}_z * (\partial_\nu \Psi_1))(x), \mathcal{R}_2(\mathcal{P}_z * (\partial_\nu \Psi_1))(x)\big)$$

where $\mathcal{R}_i$ is the i^{th} Riesz transform. Using the boundedness of the Riesz transforms on Besov spaces (part (1) of Proposition A.0.3) and the above observations regarding the Poisson kernel, we have that $\nabla\Psi_1 \in L^\infty([0,T] \times [0,\infty); \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. Recalling (4) of Proposition 2.2.2, which gives that $\nabla\Psi_2 \in L^\infty([0,T] \times [0,\infty); \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$, we have shown the following:

Theorem 2.4.3. *Let Ψ be a strong solution to $(\mathfrak{QG})$ on $[0,T]$; then there exists C depending only on $||\Psi_0||_{H^3(\mathbb{R}^3_+)}$ such that Ψ satisfies*

$$\nabla\Psi \in L^\infty([0,T]\times[0,\infty); \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$$

with norm less than or equal to C.

2.5 Propagation of Regularity

We begin by using the transport equations on both $\nabla\Psi$ and $\Delta\Psi$ to show that smoothness in the flat variable $x = (x_1, x_2)$ is propagated in time. Then, using this result in conjunction with the stratification of the flow will show that smoothness in all variables is propagated in time. Since the local existence theorem gives existence of strong solutions on a time interval which depends only on $||\nabla\Psi_0||_{H^3(\mathbb{R}^3_+)}$, obtaining a differential inequality which bounds $||\nabla\Psi(t)||_{H^3(\mathbb{R}^3_+)}$ in time allows us to apply a continuation principle, thus showing that solutions are smooth for all time. We work again on a time interval for which $\nabla\Psi$ is a solution to (QG), justifying the calculations.

Lemma 2.5.1. *For any $T > 0$, $R > 0$, there exists a constant $C_{T,R}$ such that the following is true. Let $\nabla\Psi \in L^\infty([0,t_0]; H^3(\mathbb{R}^3_+))$ be a solution to $(\mathfrak{QG})$ for all $t_0 < T$. If $||\nabla\Psi_0||_{H^{s+1}(\mathbb{R}^3_+)} < R$, then for all $t < T$,*

$$||\overline{\nabla}^{s+1}(\nabla\Psi)(t)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^{s}(\Delta\Psi)(t)||_{L^2(\mathbb{R}^3_+)} \leq C_{T,R}.$$

Proof. Recall that for $s = |\alpha|$, Proposition A.0.8 gives the commutator estimate

$$||D^\alpha(fg) - fD^\alpha g||_{L^2} \leq C(s)\left(||\nabla f||_{L^\infty}||\nabla^{(s-1)}g||_{L^2} + ||g||_{L^\infty}||\nabla^s f||_{L^2}\right).$$

Also recall that for $h = \overline{\nabla}H$, Proposition A.0.7 provides the bound

$$||h||_{L^\infty} \leq C||H||_{L^\infty} + C||h||_{\mathring{B}^0_{\infty,\infty}}\left(1 + \log\frac{||h||_{\mathring{H}^{\frac{3}{2}}}}{||h||_{\mathring{B}^0_{\infty,\infty}}}\right).$$

Using the fact that $\partial_{zz}\Psi = \Delta\Psi - \overline{\Delta}\Psi$ and applying Lemma A.0.1 with $u = \overline{\nabla}^2(\nabla\Psi)$ gives that

$$\sup_z ||\nabla\Psi(z,\cdot)||_{\mathring{H}^{\frac{5}{2}}} = \sup_z ||\overline{\nabla}^2(\nabla\Psi)(z,\cdot)||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq ||\overline{\nabla}^3(\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}. \tag{2.12}$$

From Theorem 2.4.3, $\nabla\Psi \in L^\infty([0,t_0]\times[0,\infty);\mathring{\dot{B}}^1_{\infty,\infty}(\mathbb{R}^2))$. We have that $\nabla\Psi \in L^\infty(\mathbb{R}^3_+)$. Then applying Proposition A.0.7 to $h = \overline{\nabla}(\nabla\Psi)$, Proposition 2.2.2, Theorem 2.4.3, and (2.12), we obtain the following:

$$\begin{aligned}
||\overline{\nabla}(\nabla\Psi)||_{L^\infty(\mathbb{R}^3_+)} &= \sup_z ||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{L^\infty(\mathbb{R}^2)} \\
&\le C\sup_z \left(||\nabla\Psi(z,\cdot)||_{L^\infty} + ||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{\mathring{\dot{B}}^0_{\infty,\infty}} \left(1+\log\frac{||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{\mathring{\dot{H}}^{\frac{3}{2}}}}{||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{\mathring{\dot{B}}^0_{\infty,\infty}}}\right)\right) \\
&\le C\sup_z \left(1+||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{\mathring{\dot{B}}^0_{\infty,\infty}}\left(\log||\nabla\Psi(z,\cdot)||_{\mathring{\dot{H}}^{\frac{5}{2}}} - \log||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{\mathring{\dot{B}}^0_{\infty,\infty}}\right)\right) \\
&\le C\sup_z \left(1+\log_+||\nabla\Psi(z,\cdot)||_{\mathring{\dot{H}}^{\frac{5}{2}}}\right) \\
&\le C\left(1+\log_+\left(||\overline{\nabla}^3(\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}\right)\right).
\end{aligned} \tag{2.13}$$

Here $\log_+$ denotes the standard positive part of the logarithm. We shall obtain a differential inequality from the transport equations on $\nabla\Psi$ and $\Delta\Psi$. Beginning with the former, we have from Proposition A.0.10 that

$$\partial_t(\nabla\Psi) + \mathbb{P}_\nabla(\overline{\nabla}^\perp\Psi\cdot\overline{\nabla}(\nabla\Psi)) = \nabla F.$$

We shall apply the commutator bound by putting $f = \overline{\nabla}^\perp\Psi$, $g = \overline{\nabla}(\nabla\Psi)$, and applying a differential operator $\overline{D}^\alpha$ with $|\alpha| = s+1$. Using (2.13) and the fact that $|s|\ge 2$, we have

$$\begin{aligned}
\big|\big|\big[\overline{\nabla}^\perp\Psi,\overline{D}^\alpha\big](\overline{\nabla}(\nabla\Psi)(z,\cdot))\big|\big|_{L^2(\mathbb{R}^2)} &\le C\Big(||\overline{\nabla}(\overline{\nabla}^\perp\Psi)(z,\cdot)||_{L^\infty}||\overline{\nabla}^s(\overline{\nabla}(\nabla\Psi))(z,\cdot)||_{L^2} \\
&\qquad + ||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{L^\infty}||\overline{\nabla}^{s+1}(\overline{\nabla}^\perp\Psi)(z,\cdot)||_{L^2}\Big) \\
&\le C||\overline{\nabla}^{s+1}(\nabla\Psi)(z,\cdot)||_{L^2}||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{L^\infty} \\
&\le C||\overline{\nabla}^{s+1}(\nabla\Psi)(z,\cdot)||_{L^2}\left(1+\log_+\left(||\overline{\nabla}^3(\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}\right)\right).
\end{aligned}$$

Applying the differential operator $\overline{D}^\alpha$ with $|\alpha| = s+1 \ge 3$, multiplying by $\overline{D}^\alpha\nabla\Psi$, integrating by parts, and utilizing the commutator estimate gives

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3_+} |\overline{D}^\alpha \nabla\Psi|^2 &= \int_{\mathbb{R}^3_+} \mathbb{P}_\nabla \left[[\overline{\nabla}^\perp \Psi, \overline{D}^\alpha](\overline{\nabla}(\nabla\Psi)) \right] \cdot \nabla \overline{D}^\alpha \Psi + \int_{\mathbb{R}^3_+} \nabla \overline{D}^\alpha F \cdot \nabla \overline{D}^\alpha \Psi \\
&= \int_{\mathbb{R}^3_+} \left[\overline{\nabla}^\perp \Psi, \overline{D}^\alpha \right] (\overline{\nabla}(\nabla\Psi)) \cdot \nabla \overline{D}^\alpha \Psi + \int_{\mathbb{R}^2} (\overline{D}^\alpha(\partial_\nu F))(\overline{D}^\alpha \Psi) \\
&= \int_{\mathbb{R}^3_+} \left[\overline{\nabla}^\perp \Psi, \overline{D}^\alpha \right] (\overline{\nabla}(\nabla\Psi)) \cdot \nabla \overline{D}^\alpha \Psi + \int_{\mathbb{R}^2} (\overline{D}^\alpha(\overline{\Delta}\Psi))(\overline{D}^\alpha \Psi) \\
&\leq \int_0^\infty \int_{\mathbb{R}^2} \left[\overline{\nabla}^\perp \Psi(z,\cdot), \overline{D}^\alpha \right] (\overline{\nabla}(\nabla\Psi)(z,\cdot)) \cdot \nabla \overline{D}^\alpha \Psi(z,\cdot)\, dx\, dz \\
&\leq \int_0^\infty \left|\left| \left[\overline{\nabla}^\perp \Psi(z,\cdot), \overline{D}^\alpha \right] (\overline{\nabla}(\nabla\Psi)(z,\cdot)) \right|\right|_{L^2(\mathbb{R}^2)} \left|\left| \overline{D}^\alpha \nabla\Psi(z,\cdot) \right|\right|_{L^2(\mathbb{R}^2)} dz \\
&\leq C \int_0^\infty \left|\left| \overline{\nabla}^{s+1} (\nabla\Psi)(z,\cdot) \right|\right|^2_{L^2(\mathbb{R}^2)} \left(1 + \log_+ \left(||\overline{\nabla}^3 (\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2 (\Delta\Psi)||_{L^2(\mathbb{R}^3_+)} \right) \right) dz \\
&\leq C \left|\left| \overline{\nabla}^{s+1} (\nabla\Psi) \right|\right|^2_{L^2(\mathbb{R}^3_+)} \left(1 + \log_+ \left(||\overline{\nabla}^3 (\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2 (\Delta\Psi)||_{L^2(\mathbb{R}^3_+)} \right) \right).
\end{aligned}
$$

We now move to the transport equation on $\Delta\Psi$:

$$\partial_t(\Delta\Psi) + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\Delta\Psi) = 0.$$

We shall apply the commutator bound by putting $f = \overline{\nabla}^\perp \Psi$, $g = \Delta\Psi$, and applying a differential operator $\overline{D}^\alpha$ with $|\alpha| = s$. Using the L^∞ bound on $\Delta\Psi$, (2.13), and the fact that $|s| \geq 2$, we have

$$
\begin{aligned}
\left|\left| [\overline{\nabla}^\perp \Psi, \overline{D}^\alpha \overline{\nabla} \cdot\,](\Delta\Psi)(z,\cdot) \right|\right|_{L^2(\mathbb{R}^2)} &\leq C \Big(||\overline{\nabla}(\overline{\nabla}^\perp \Psi)(z,\cdot)||_{L^\infty} ||\overline{\nabla}^s (\Delta\Psi)(z,\cdot)||_{L^2} \\
&\qquad + ||\Delta\Psi(z,\cdot)||_{L^\infty} ||\overline{\nabla}^{s+1} (\overline{\nabla}^\perp \Psi)(z,\cdot)||_{L^2} \Big) \\
&\leq C \left(||\overline{\nabla}^s (\Delta\Psi)(z,\cdot)||_{L^2} ||\overline{\nabla}(\nabla\Psi)(z,\cdot)||_{L^\infty} + ||\overline{\nabla}^{s+1} (\overline{\nabla}^\perp \Psi)(z,\cdot)||_{L^2} \right) \\
&\leq C \left(||\overline{\nabla}^s (\Delta\Psi)(z,\cdot)||_{L^2} + ||\overline{\nabla}^{s+1} (\nabla\Psi)(z,\cdot)||_{L^2} \right) \\
&\qquad \times \left(1 + \log_+ \left(||\overline{\nabla}^3 (\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2 (\Delta\Psi)||_{L^2(\mathbb{R}^3_+)} \right) \right).
\end{aligned}
$$

Applying the differential operator $\overline{D}^\alpha$ with $|\alpha| = s \geq 2$, multiplying by $\overline{D}^\alpha \Delta\Psi$, integrating by parts, and utilizing the commutator estimate gives

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3_+} |\overline{D}^\alpha \Delta\Psi|^2 &= \int_{\mathbb{R}^3_+} \left[\overline{\nabla}^\perp \Psi, \overline{D}^\alpha \overline{\nabla}\cdot\right](\Delta\Psi)\cdot \overline{D}^\alpha \Delta\Psi \\
&= \int_0^\infty \int_{\mathbb{R}^2} \left[\overline{\nabla}^\perp \Psi(z,x), \overline{D}^\alpha \overline{\nabla}\cdot\right](\Delta\Psi)(z,x)\cdot \overline{D}^\alpha \Delta\Psi(z,x)\, dx\, dz \\
&\leq \int_0^\infty \left|\left|\left[\overline{\nabla}^\perp \Psi(z,\cdot), \overline{D}^\alpha \overline{\nabla}\cdot\right](\Delta\Psi)(z,\cdot)\right|\right|_{L^2(\mathbb{R}^2)} \left|\left|\overline{D}^\alpha \Delta\Psi(z,\cdot)\right|\right|_{L^2(\mathbb{R}^2)} dz \\
&\leq C\int_0^\infty \left(||\overline{\nabla}^s(\Delta\Psi)(z,\cdot)||^2_{L^2(\mathbb{R}^2)} + ||\overline{\nabla}^{s+1}(\nabla\Psi)(z,\cdot)||_{L^2(\mathbb{R}^2)}||\overline{\nabla}^s(\Delta\Psi)(z,\cdot)||_{L^2(\mathbb{R}^2)}\right) \\
&\qquad\qquad \times\left(1+\log_+ ||\overline{\nabla}^3(\nabla\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}\right) dz \\
&\leq C\left(\left|\left|\overline{\nabla}^s(\Delta\Psi)\right|\right|^2_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^{s+1}(\nabla\Psi)||_{L^2(\mathbb{R}^3_+)}||\overline{\nabla}^s(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}\right) \\
&\qquad \times\left(1+\log_+\left(||\overline{\nabla}^3(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^2(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)}\right)\right).
\end{aligned}
$$

Therefore, we can sum over α in both inequalities and apply Gronwall's inequality to the sum

$$||\overline{\nabla}^{s+1}(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)} + ||\overline{\nabla}^s(\Delta\Psi)||_{L^2(\mathbb{R}^3_+)},$$

finishing the proof. □

We now show that regularity in z can be propagated as well.

Theorem 2.5.2. *For any $T > 0$, $R > 0$, there exists a constant $C_{T,R}$ such that the following is true. Let $\nabla\Psi \in L^\infty([0,t_0]; H^3(\mathbb{R}^3_+))$ be a solution to $(\mathcal{QG})$ for all $t_0 < T$. If $||\nabla\Psi_0||_{H^s(\mathbb{R}^3_+)} < R$, then for all $t < T$,*

$$||\nabla\Psi(t)||_{H^s(\mathbb{R}^3_+)} \leq C_{T,R}.$$

Proof. From Lemma 2.5.1, Sobolev embedding, and the trace estimate, $||\overline{\nabla}(\nabla\Psi)(t)||_{L^\infty(\mathbb{R}^3_+)}$ is bounded. Also, observe that using the identity $\partial_{zz} = \Delta - \overline{\Delta}$, we have that

$$||\nabla^s(\nabla\Psi)||_{L^2} \leq C\left(||\nabla^{s-1}(\Delta\Psi)||_{L^2} + ||\overline{\nabla}^s(\nabla\Psi)||_{L^2}\right).$$

By Lemma 2.5.1, we have that $||\overline{\nabla}^s(\nabla\Psi)||_{L^2} < \infty$. Thus the theorem will be shown if $\Delta\Psi \in H^{s-1}$ for all time. Applying a differential operator D^α with $|\alpha| = s-1 \geq 2$, multiplying by $D^\alpha \Delta\Psi$, integrating by parts, and using the commutator estimate (in $\mathbb{R}^3_+$) in conjunction with the above observations, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3_+} |D^\alpha \Delta\Psi|^2 &= \int_{\mathbb{R}^3_+} \left[\overline{\nabla}^\perp \Psi, D^\alpha \overline{\nabla}\cdot\right](\Delta\Psi)\cdot D^\alpha \Delta\Psi \\
&\leq C\left(||\overline{\nabla}(\nabla\Psi)||_{L^\infty}||\nabla^{s-1}(\Delta\Psi)||_{L^2} + ||\Delta\Psi||_{L^\infty}||\nabla^s(\nabla\Psi)||_{L^2}\right)||\nabla^{s-1}(\Delta\Psi)||_{L^2} \\
&\leq C(||\nabla^{s-1}(\Delta\Psi)||^2_{L^2} + ||\nabla^{s-1}(\Delta\Psi)||_{L^2})
\end{aligned}$$

Summing over α and applying Gronwall's inequality now finishes the proof. $\square$

Proof of Theorem 2.1. We have showed that if $\nabla\Psi_0 \in H^s(\mathbb{R}^3_+)$ for some $s \geq 3$, then for all $T > 0$, there exists $C(T,s)$ such that for all $t \leq T$, $||\nabla\Psi(t,\cdot)||_{H^s(\mathbb{R}^3_+)} \leq C(T,s)$. We apply a continuation principle argument in conjunction with Theorem 2.5.2 to prove the first part of Theorem 2.1. Since the time of existence in Proposition 2.2.1 depends only on the H^3 norm of $\nabla\Psi$, a quantity which satisfies a differential equality thanks to Theorem 2.5.2, we can repeatedly apply the local existence result to obtain a global in time classical solution. To finish the proof, it remains to show uniqueness and regularity in time. Uniqueness follows from the usual energy method. Indeed, let Ψ_1, Ψ_2 be two solutions with the same initial data $\nabla\Psi_0 \in H^s(\mathbb{R}^3_+)$ for some $s \geq 3$. We will use the formulation of Proposition A.0.10 with $\tilde{\Psi} = \Psi_1 - \Psi_2$, $\tilde{F} = F_1 - F_2$. Considering the difference of the two equations, we have

$$\partial_t(\nabla\tilde{\Psi}) + \overline{\nabla}^\perp\tilde{\Psi}\cdot\overline{\nabla}(\nabla\Psi_1) + \overline{\nabla}^\perp\Psi_2\cdot\overline{\nabla}(\nabla\tilde{\Psi}) = \nabla\tilde{F}.$$

Multiplying by $\nabla\tilde{\Psi}$, using the regularity of $\nabla\Psi_1$, and integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}||\nabla\tilde{\Psi}||^2_{L^2} &= \int_{\mathbb{R}^3_+}\left(\overline{\nabla}^\perp\tilde{\Psi}\cdot\overline{\nabla}(\nabla\Psi_1) + \overline{\nabla}^\perp\Psi_2\cdot\overline{\nabla}(\nabla\tilde{\Psi})\right)\cdot\nabla\tilde{\Psi} + \int_{\mathbb{R}^3_+}\nabla\tilde{F}\cdot\nabla\tilde{\Psi} \\
&= \int_{\mathbb{R}^3_+}\left(\overline{\nabla}^\perp\tilde{\Psi}\cdot\overline{\nabla}(\nabla\Psi_1)\right)\cdot\nabla\tilde{\Psi} + \int_{\mathbb{R}^2}\overline{\Delta}\tilde{\Psi}\tilde{\Psi} \\
&\leq C||\nabla\tilde{\Psi}||^2_{L^2}.
\end{aligned}$$

Since $\nabla\tilde{\Psi}|_{t=0} = 0$, Gronwall's inequality shows that $\nabla\tilde{\Psi} = 0$ for all time. For the regularity in space and time, now assume that Ψ is a solution to $(\mathcal{QG})$ with smooth initial data. Using the equalities

$$\partial_t(\Delta\Psi) = -\overline{\nabla}^\perp\Psi\cdot\overline{\nabla}(\Delta\Psi)$$

$$\partial_t(\partial_\nu \Psi) = -\overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\partial_\nu \Psi) + \overline{\Delta}\Psi$$

and noticing that Theorem 2.5.2 gives that any spatial derivative of the right hand side in either equality is bounded, we have that $\Delta\Psi$, $\partial_\nu \Psi$ and all their spatial derivatives are C^1 in time. Differentiating the equations in time and continuing inductively finishes the proof of Theorem 2.1. $\square$

Chapter 3

Existence of Weak Solutions for the Inviscid Model in $\mathbb{R}^3_+$

3.1 Overview

In this chapter, we study the 3-D inviscid quasi-geostrophic system $(QG)_I$ posed in the upper half-space $\mathbb{R}^3_+$

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^\perp\Psi\cdot\overline{\nabla}(\Delta\Psi) = f_L & t>0,\quad z>0,\quad x=(x_1,x_2)\in\mathbb{R}^2 \qquad (\mathrm{QG})_L \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^\perp\Psi\cdot\overline{\nabla}(\partial_\nu\Psi) = f_\nu & t>0,\quad z=0,\quad x=(x_1,x_2)\in\mathbb{R}^2 \qquad (\mathrm{QG})_\nu \end{cases}$$

supplied with an initial data Ψ_0. As usual,

$$\Psi : \underset{(t,z,x)}{[0,\infty)\times\mathbb{R}_+\times\mathbb{R}^2} \to \underset{\Psi(t,z,x)}{\overset{}{\mathbb{R}}}$$

is the stream function for the geostrophic flow, and f_L and f_ν are forcing terms. We have again simplified the equation $(QG)_L$ by choosing the elliptic operator to simply by the Laplacian and excluding the term $\beta_0 y$. However, all of the results in this chapter should hold for more general elliptic operators $\mathcal{L}$ for which $\lambda(z)\neq 1$.

The purpose of this chapter is to study the existence and properties of various types of weak solutions to this system following the work of the author [77]. We provide global existence results for initial data belonging to Lebesgue spaces. Much mathematical research has been focused on this system and its variants. Beale and Bourgeois [7] and Desjardins and Grenier [42] derived the system from physical principles. Puel and Vasseur [82] first proved the global existence of weak solutions in the case of L^2 initial data, using a projection operator to reformulate the problem. Recall that when $\Delta\Psi_0\equiv 0$ and there are no forcing terms, $\Psi(t)$ remains harmonic for all time t. Then using that $(-\overline{\Delta})^{\frac12}$ is the Dirichlet-to-Neumann operator for $\mathbb{R}^3_+$, we set for each time $t\geq 0$

$$\theta := \partial_\nu\Psi = (-\overline{\Delta})^{\frac12}\Psi,\qquad u := \overline{\nabla}^\perp\Psi = \Big(0, -\mathcal{R}_2(-\overline{\Delta})^{\frac12}\Psi, \mathcal{R}_1(-\overline{\Delta})^{\frac12}\Psi\Big) = \mathcal{R}^\perp\theta,$$

where $\mathcal{R}_1, \mathcal{R}_2$ are the Riesz transforms in $\mathbb{R}^2$. Then (QG) reduces to the well-studied inviscid surface quasi-geostrophic equation, which can be written as

$$\partial_t \theta + u \cdot \overline{\nabla} \theta = 0.$$

For the sake of consistency and to keep in mind the connection to the 3D model, we shall always treat $\overline{\nabla}, \overline{\nabla}^{\perp}$, and $\mathcal{R}^{\perp}$ as vectors with three components and zero first component. SQG has received considerable attention due to its similarities with the important systems of fluid mechanics (see Constantin, Majda, and Tabak [27], Garner, Held, Pierrehumber, and Swanson [54], among others). Weak solutions were constructed in L^2 by Resnick [83]. Marchand [71] first gave a proof of the existence of global weak solutions when the initial data is not in L^2 but rather L^p for any $p > \frac{4}{3}$.

3.1.1 The Reformulated Problem

A crucial tool in our analysis will be a reformulation of (QG). We draw inspiration from Puel and Vasseur [82], who used a reformulation to obtain their global existence result. The physical system as written is analogous to the vorticity form of the Euler equations with an additional boundary condition. However, one may consider the following reformulation, in which $\mathrm{curl}(Q)$ acts as a Lagrange multiplier similar to the gradient of the pressure in the Euler equations:

$$\begin{cases} \partial_t(\nabla\Psi) + \overline{\nabla}^{\perp}\Psi : \overline{\nabla}(\nabla\Psi) = \mathrm{curl}\, Q + \nabla F & z > 0 \\ \mathrm{curl}\, Q \cdot \nu = 0, \quad \partial_\nu F = f_\nu & z = 0 \\ \Delta F = f_L & z > 0. \end{cases} \qquad \text{(rQG)}$$

Formally, taking the divergence of (rQG) gives $(QG)_L$, and taking the trace gives $(QG)_\nu$. To obtain (rQG) from (QG), one must invert the divergence operator coupled with a Neumann boundary condition. While providing a link between the two formulations will be an important part of our analysis (see Theorem 3.2), let us proceed from the perspective of (rQG) for the time being. Following Puel and Vasseur [82], we define the notion of weak solutions to (rQG).

Definition 3.1.1 (Weak Solutions to (rQG)). *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T,T) \times (-R,R)^3$, and F be such that $\Delta F = f_L$, $\partial_\nu F = f_\nu$. A weak solution Ψ to (rQG) with forcing f_ν, f_L on $(0,T) \times \mathbb{R}^3_+$ must satisfy*

$$
\begin{aligned}
-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi \right) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F \right) dx\, dz\, dt \\
= \int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0,z,x) \cdot \nabla \Psi(0,z,x)\, dx\, dz
\end{aligned}
$$

for all R, ϕ. For the weak formulation to make sense, we require $\nabla \Psi, \overline{\nabla}^\perp \Psi \otimes \nabla \Psi \in L^1_{loc}([0,T] \times \mathbb{R}^3_+)$.

We remark that the definition of weak solutions contains no information about curl(Q). Indeed, the choice of test functions formally encodes the fact that inverting the divergence operator is unique only up to the curl of a vector field. We shall prove the global existence of weak solutions to (rQG).

Theorem 3.1. *Suppose that $p \in (\frac{4}{3}, \infty]$ and $q \in (\frac{6}{5}, 3]$. Let $\theta \in L^p(\mathbb{R}^2)$, $\omega \in L^q(\mathbb{R}^3_+)$, $f_L \in L^1\left([0,T]; L^{\frac{6}{5}} \cap L^q(\mathbb{R}^3_+)\right)$, and $f_\nu \in L^1\left([0,T]; L^{\frac{4}{3}} \cap L^p(\mathbb{R}^2)\right)$ for all $T > 0$. When $p = \infty$ we additionally require a finite p' such that $\theta \in L^{p'}(\mathbb{R}^2)$, and when $q = 3$ we additionally require a $q' \in (\frac{6}{5}, 3)$ such that $\omega \in L^{q'}(\mathbb{R}^3_+)$. Then there exists a global weak solution $\nabla \Psi$ on $(0,\infty) \times \mathbb{R}^3_+$ to (rQG) with forcing f_ν, f_L such that $\Delta \Psi|_{t=0} = \omega$ and $\partial_\nu \Psi|_{t=0} = \theta$. In addition, there exists a constant C such that for all $T > 0$, Ψ satisfies the following bound:*

$$
\begin{aligned}
&\|\nabla \Psi\|_{L^\infty\left([0,T]; L^{\frac{3p}{2}}(\mathbb{R}^3_+) + L^{\frac{3q}{3-q}}(\mathbb{R}^3_+)\right)} + \|\Delta \Psi\|_{L^\infty\left([0,T]; L^q(\mathbb{R}^3_+)\right)} + \|\partial_\nu \Psi\|_{L^\infty([0,T]; L^p(\mathbb{R}^2))} \\
&\quad \le C \left(\|\omega\|_{L^q} + \|\theta\|_{L^p} + \|f_L\|_{L^1\left([0,T]; L^q(\mathbb{R}^3_+)\right)} + \|f_\nu\|_{L^1([0,T]; L^p(\mathbb{R}^2))} \right).
\end{aligned}
$$

Let us give a simple explanation for the restrictions on p and q. In order for the nonlinear term $\overline{\nabla} \cdot (\overline{\nabla}^\perp \Psi \otimes \nabla \Psi)$ to be well-defined as a distribution from integration by parts, we need $\nabla \Psi \in L^2(\mathbb{R}^3_+)$ (at least locally). If $\Delta \Psi_0 \in L^{\frac{6}{5}}(\mathbb{R}^3_+)$ and $\partial_\nu \Psi_0 \in L^{\frac{4}{3}}(\mathbb{R}^2)$, then solving the elliptic boundary value problem gives $\nabla\Psi_0 \in L^2(\mathbb{R}^3_+)$, hence the restrictions on q and p. If $q = 3$ or $p = \infty$, the corresponding Lebesgue norm on $\nabla\Psi$ is actually the standard BMO norm in the space of functions of bounded mean oscillation; for simplicity's sake we employ this abbreviation. The additional assumptions on θ when $p = \infty$ and ω when $q = 3$ are technical requirements which are necessary to handle the decay at infinity of functions defined in $\mathbb{R}^3_+$. The solutions we construct are obtained by taking a weak limit of smooth solutions to a regularized system. Global smooth solutions for the regularized system are constructed following [79]. We refer to the preliminaries for a precise statement of the result we shall use, and Appendix B for a brief description of the techniques.

3.1.2 Different Notions of Weak Solutions

It is interesting to consider whether weak solutions to (rQG) might be weak solutions to (QG), and vice versa. In this section we address this question, therein justifying our use of the reformulated system. We define two classes of weak solutions to (QG); the first is the more standard notion of weak solution, while the second incorporates the Calderón commutator used in the existence proofs of Marchand [71] and Resnick [83].

Definition 3.1.2 (Weak Solutions to (QG)). *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T,T)\times(0,R)\times(-R,R)^2$, and $\bar{\phi} \in C^\infty(\mathbb{R}^3)$ compactly supported in $(-T,T)\times(-R,R)^2$. A weak solution Ψ to (QG) on $(0,T)\times\mathbb{R}^3_+$ with forcing f_ν, f_L must satisfy*

$$\begin{aligned} -\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\Big(\Big(\partial_t\phi + \overline{\nabla}^\perp\Psi\cdot\overline{\nabla}\phi\Big)\Delta\Psi + \phi f_L\Big)\,dx\,dz\,dt \\ = \int_0^\infty\int_{\mathbb{R}^2}\phi(0,z,x)\Delta\Psi(0,z,x)\,dx\,dz \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -\int_0^T\int_{\mathbb{R}^2}\Big(\Big(\partial_t\bar{\phi} + \overline{\nabla}^\perp\Psi(t,0,x)\cdot\overline{\nabla}\bar{\phi}\Big)\partial_\nu\Psi(t,x) + \bar{\phi}f_\nu\Big)\,dx\,dt \\ = \int_{\mathbb{R}^2}\bar{\phi}(0,x)\partial_\nu\Psi(0,x)\,dx \end{aligned} \tag{3.2}$$

for all $R, \phi, \bar{\phi}$. For the weak formulation to make sense, we require $\Delta\Psi, \overline{\nabla}^{\perp}\Psi\Delta\Psi \in L^1_{loc}([0,T]\times\mathbb{R}^3_+)$ and $\partial_\nu\Psi, \overline{\nabla}^{\perp}\Psi\partial_\nu\Psi \in L^1_{loc}([0,T]\times\mathbb{R}^2)$.

For functions of two variables, $\bar{\Lambda}\theta = \sqrt{-\bar{\Delta}}(\theta)$ and $\bar{\Lambda}^{-1}$ is the corresponding inverse operator. In addition,

$$\mathcal{R}^{\perp}\theta = (0, -\mathcal{R}_2\theta, \mathcal{R}_1\theta)$$

is the rotated vector of Riesz transforms with zero first component as usual. The commutator $[A,B]$ of two operators is $AB-BA$. In the following definition, we use the commutator result of Marchand [71] to define a notion of weak solution for (QG) for low levels of integrability. Marchand's results concerning boundedness and convergence of the commutator are stated in the preliminaries. For the sake of brevity we suppress for now issues concerning the frequency support of $\partial_\nu\Psi$; these are also addressed in the preliminaries.

Definition 3.1.3 (**Weak Solutions to** (**QG**) **with Commutator**)**.** *Let T, R be fixed, $\phi \in C^\infty(\mathbb{R}^4)$ compactly supported in $(-T,T)\times(0,R)\times(-R,R)^2$, and $\bar{\phi}\in C^\infty(\mathbb{R}^3)$ compactly supported in $(-T,T)\times(-R,R)^2$. Let $\Psi : [0,T)\times\mathbb{R}^3_+ \to \mathbb{R}$ be given and Ψ_1 and Ψ_2 be defined for all $t\in[0,T)$ by the boundary value problems*

$$\begin{cases}\Delta\Psi_1 = 0\\ \partial_\nu\Psi_1 = \partial_\nu\Psi\end{cases}\qquad\begin{cases}\Delta\Psi_2 = \Delta\Psi\\ \partial_\nu\Psi_2 = 0.\end{cases}$$

We define $\left(\overline{\nabla}^{\perp}\Psi(t,0,x)\partial_\nu\Psi(t,x)\right)_C$ as a distribution by (and use the notation $(\cdot)_C$ to specify that we are using the commutator formulation)

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2}\left(\overline{\nabla}^{\perp}\Psi(t,0,x)\partial_\nu\Psi(t,x)\right)_C\cdot\overline{\nabla}\bar{\phi}\,dx\,dt &:= \frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\left(\mathcal{R}^{\perp}(\partial_\nu\Psi_1)\right)\cdot\left([\bar{\Lambda},\overline{\nabla}\bar{\phi}](\bar{\Lambda}^{-1}\partial_\nu\Psi_1)\right)dx\,dt\\ &\quad+\int_0^T\int_{\mathbb{R}^2}\left(\overline{\nabla}^{\perp}\Psi_2(t,0,x)\partial_\nu\Psi_1(t,x)\right)\cdot\overline{\nabla}\bar{\phi}\,dx\,dt\\ &= \frac{1}{2}\int_0^T\int_{\mathbb{R}^2}\left(0,-\mathcal{R}_2(\partial_\nu\Psi_1),\mathcal{R}_1(\partial_\nu\Psi_1)\right)\cdot\left(\bar{\Lambda}\big(\overline{\nabla}\phi\bar{\Lambda}^{-1}(\partial_\nu\Psi_1)\big)-\overline{\nabla}\phi\partial_\nu\Psi_1\right)dx\,dt\\ &\quad+\int_0^T\int_{\mathbb{R}^2}\left(\overline{\nabla}^{\perp}\Psi_2(t,0,x)\partial_\nu\Psi_1(t,x)\right)\cdot\overline{\nabla}\bar{\phi}\,dx\,dt\end{aligned}$$

and say that Ψ is a weak solution to (QG) with commutator on $(0,T)\times\mathbb{R}^3_+$ with forcing f_ν,

f_L if

$$-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla} \phi \right) \Delta \Psi + \phi f_L \right) \, dx \, dz \, dt$$
$$= \int_0^\infty \int_{\mathbb{R}^2} \phi(0,z,x) \Delta \Psi(0,z,x) \, dx \, dz$$

and

$$-\int_0^T \int_{\mathbb{R}^2} \left(\partial_t \bar{\phi} \partial_\nu \Psi + \left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi \right)_C \cdot \overline{\nabla} \bar{\phi} + \bar{\phi} f_\nu \right) \, dx \, dt$$
$$= \int_{\mathbb{R}^2} \bar{\phi}(0,x) \partial_\nu \Psi(0,x) \, dx$$

for all $T, R, \phi, \bar{\phi}$. For the weak formulation to make sense, we require $\partial_\nu \Psi(t) \in L^p(\mathbb{R}^2)$ for all time t and some $p \in (\frac{4}{3}, 2]$ and $\overline{\nabla}^\perp \Psi_2 \partial_\nu \Psi \in L^1_{loc}([0,T] \times \mathbb{R}^2)$.

We can now connect the weak solutions of Definition 3.1.1, Definition 3.1.2, and Definition 3.1.3.

Theorem 3.2. *1. Assume that $\Delta \Psi \in L^\infty \left([0,T); L^q(\mathbb{R}^3_+) \right)$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu \Psi \in L^\infty \left([0,T); L^p(\mathbb{R}^2) \right)$ for $p \in [2, \infty]$. Then $\nabla \Psi$ satisfies Definition 3.1.1 if and only if $\nabla \Psi$ satisfies Definition 3.1.2.*

2. Assume that $\Delta \Psi \in L^\infty \left([0,T); L^q(\mathbb{R}^3_+) \right)$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu \Psi \in L^\infty \left([0,T); L^p(\mathbb{R}^2) \right)$ for $p \in (\frac{4}{3}, 2]$. Assume in addition that

$$p \geq \frac{2q}{3(q-1)}.$$

Then $\nabla \Psi$ satisfies Definition 3.1.1 if and only if $\nabla \Psi$ satisfies Definition 3.1.3.

3. Assume that $\Delta \Psi \in L^\infty \left([0,T); L^q(\mathbb{R}^3_+) \right)$ for $q \in [\frac{3}{2}, 3]$ and $\partial_\nu \Psi \in L^\infty \left([0,T); L^p \cap L^r(\mathbb{R}^2) \right)$ for $p \in (\frac{4}{3}, 2]$, $r \in [2, \infty]$. Then $\nabla \Psi$ satisfies Definition 3.1.2 if and only if $\nabla \Psi$ satisfies Definition 3.1.3.

Theorem 3.2 complements the existence result in Theorem 3.1. Indeed, imposing that the initial data $\nabla \Psi_0$, $\Delta \Psi_0$, and $\partial_\nu \Psi_0$ all belong to L^2, then we recover the result of Puel and Vasseur [82]. Imposing $\Delta\Psi_0 \equiv 0$ and $\partial_\nu \Psi_0 \in L^p(\mathbb{R}^2)$, we recover the result of Marchand [71].

It is interesting to note that if the initial data satisfies $\Delta\Psi_0 \in L^{\frac{6}{5}}(\mathbb{R}^3_+)$ and $\partial_\nu \Psi_0 \equiv 0$ to remove the boundary condition, trace theory would give $\overline{\nabla}^\perp \Psi_0|_{z=0} \in L^{\frac{4}{3}}(\mathbb{R}^2)$ (see Lemma A.0.1), corresponding precisely to the lower limit of integrability in the proof of Marchand. Conversely, imposing that $\Delta\Psi_0 \equiv 0$ and $\partial_\nu \Psi_0 \in L^{\frac{4}{3}}(\mathbb{R}^2)$ to eliminate the transport equation for $z > 0$, Lemma 3.2.4 ensures that $\nabla\Psi_0 \in L^2(\mathbb{R}^3_+)$. In addition, one can see from the proof of Theorem 3.2 that

$$p \geq \frac{2q}{3(q-1)}$$

is the minimum integrability needed to define the nonlinear terms in both $(QG)_L$ and $(QG)_\nu$. Thus, the conditions on p and q correspond in a natural way and appear to be the sharpest possible afforded by the structure of the system. Furthermore, our analysis combines the reformulation (rQG) of Vasseur and Puel and the commutator of Marchand. In conjunction with the correspondence between the conditions on p and q, this naturally connects the two approaches.

3.2 Preliminaries

The following theorem modifies the statement of Theorem 2.1 to allow for forcing terms. We provide an outline of the minor modifications necessary in Appendix (B).

Theorem 3.2.1 (Regularized System). *Consider the regularized system $(QG)_\epsilon$*

$$\begin{cases} \partial_t(\Delta\Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\Delta\Psi_\epsilon) = f_{L,\epsilon} & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu \Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\partial_\nu \Psi_\epsilon) = f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}}(\partial_\nu \Psi_\epsilon) & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

supplied with initial data $\Delta\Psi_{0,\epsilon}$, $\partial_\nu \Psi_{0,\epsilon}$ which are C^∞ and compactly supported. Suppose that $f_{L,\epsilon} \in L^1\left([0,T]; L^1 \cap L^q(\mathbb{R}^3_+)\right) \cap L^\infty([0,T]; C^k(\mathbb{R}^3_+))$, $f_{\nu,\epsilon} \in L^1\left([0,T]; L^1 \cap L^p(\mathbb{R}^2)\right) \cap L^\infty([0,T]; C^k(\mathbb{R}^2))$ for all $T > 0$, $k \in \mathbb{N}$ and that for each time, $f_{L,\epsilon}$ and $f_{\nu,\epsilon}$ have spatial support contained in $[-\frac{5}{\epsilon}, \frac{5}{\epsilon}]^3$ and $[-\frac{5}{\epsilon}, \frac{5}{\epsilon}]^2$, respectively. Then there exists a unique, global in time classical solution $\nabla\Psi_\epsilon$ and a constant C independent of ϵ such that $\nabla\Psi_\epsilon$ satisfies the energy estimates for $t \in [0,T]$

1. $\|\Delta\Psi_\epsilon(t)\|_{L^q} \leq C(\|f_{L,\epsilon}\|_{L^1([0,T];L^q)} + \|\Delta\Psi_{0,\epsilon}\|_{L^q})$

2. $\|\partial_\nu\Psi_\epsilon(t)\|_{L^p} \leq C(\|f_{\nu,\epsilon}\|_{L^1([0,T];L^p)} + \|\partial_\nu\Psi_{0,\epsilon}\|_{L^p})$

3. $\|\nabla\Psi_\epsilon(t)\|_{L^{\frac{3q}{3-q}}+L^{\frac{3p}{2}}} \leq C(\|f_{L,\epsilon}\|_{L^1([0,T];L^q)} + \|\Delta\Psi_{0,\epsilon}\|_{L^q} + \|f_{\nu,\epsilon}\|_{L^1([0,T];L^p)} + \|\partial_\nu\Psi_{0,\epsilon}\|_{L^p})$

Let us now state results of Marchand [71].

Lemma 3.2.2 (Calderón Commutator). *1. For $f \in L^p(\mathbb{R}^2)$, $p \in (\frac{4}{3}, 2]$, and $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^2)$, $\overline{\nabla}\cdot(f\mathcal{R}^\perp f)$ is defined as a distribution by*

$$\langle \phi, \overline{\nabla}\cdot(f\mathcal{R}^\perp f)\rangle := \frac{1}{2}\int_{\mathbb{R}^2}\left(\mathcal{R}^\perp f\right)\cdot\left([\bar{\Lambda},\overline{\nabla}\phi]\left(\bar{\Lambda}^{-1}f\right)\right).$$

If $f \in L^2(\mathbb{R}^2)$ is such that $\hat{f}$ is zero in a neighborhood of the origin, then

$$\int_{\mathbb{R}^2}\left(f\mathcal{R}^\perp f\right)\cdot\overline{\nabla}\phi = -\frac{1}{2}\int_{\mathbb{R}^2}\left(\mathcal{R}^\perp f\right)\cdot\left([\bar{\Lambda},\overline{\nabla}\phi]\left(\bar{\Lambda}^{-1}f\right)\right).$$

2. Let $p \in (\frac{4}{3}, \infty]$ and $\{\theta_\epsilon(t,x)\}_{\epsilon>0} \subset L^\infty([0,T];L^p(\mathbb{R}^2))$ be a sequence of functions such that θ_ϵ converges weakly- to $\theta(t,x) \in L^\infty([0,T];L^p(\mathbb{R}^2))$, T fixed. Then the following holds in the sense of distributions:*

$$\lim_{\epsilon\to 0}\overline{\nabla}\cdot(\theta_\epsilon\mathcal{R}^\perp\theta_\epsilon) = \overline{\nabla}\cdot(\theta\mathcal{R}^\perp\theta).$$

Here it is understood that for $p \geq 2$, $\overline{\nabla}\cdot(\theta_\epsilon\mathcal{R}^\perp\theta_\epsilon)$ is defined by integration by parts, whereas for $p \leq 2$, we use the commutator.

Decomposing an arbitrary function using Littlewood-Paley projections allows one to use the commutator only for the high-frequency piece. To avoid cumbersome Besov space notations, we suppress these details and will write

$$\mathcal{R}^\perp\theta[\bar{\Lambda},\overline{\nabla}\phi]\left(\bar{\Lambda}^{-1}\theta\right)$$

for any L^p function with $p \in (\frac{4}{3}, 2]$. We refer the reader to Marchand [71] for further details and proofs.

3.2.1 Elliptic Estimates

We now specify the appropriate Lebesgue spaces and obtain the corresponding bounds for the solution to the Poisson problem with Neumann boundary data in the upper half space. While the results are standard, we include proofs in Appendix B for the sake of completeness. We also include a technical lemma which will be useful in the proof of Theorem 3.1. Definitions of the function spaces we use are contained in Appendix B.

Lemma 3.2.3. *Given $f \in L^q(\mathbb{R}^3_+)$ for $q \in (1,3]$, there exists a unique $u \in \mathring{W}^{1,\frac{3q}{3-q}}(\mathbb{R}^3_+)$ ($\nabla u \in BMO$ if $q = 3$) such that*

$$\begin{cases} -\Delta u = f & z > 0 \\ \partial_\nu u = 0 & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3q}{3-q}}(\mathbb{R}^3_+)} \leq C(q)\|f\|_{L^q(\mathbb{R}^3_+)}, \qquad q < 3$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}^3_+)} \leq C(q)\|f\|_{L^q(\mathbb{R}^3_+)}, \qquad q = 3.$$

For the following lemma we use the space

$$\mathring{W}^{1,p}_\Delta(\mathbb{R}^3_+) := \{u \in \mathring{W}^{1,p}(\mathbb{R}^3_+) | \Delta u = 0 \ in \ \mathcal{D}'(\mathbb{R}^3_+)\}$$

with norm

$$\|u\|_{\mathring{W}^{1,p}_\Delta(\mathbb{R}^3_+)} = \|\nabla u\|_{L^p(\mathbb{R}^3_+)}$$

Lemma 3.2.4. *Given $g \in L^p(\mathbb{R}^2)$ for $p \in (1,\infty]$, there exists $u \in \mathring{W}^{1,\frac{3p}{2}}_\Delta(\mathbb{R}^3_+)$ solving*

$$\begin{cases} \Delta u = 0 & z > 0 \\ \partial_\nu u = g & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3p}{2}}(\mathbb{R}^3_+)} \leq C(p)\|g\|_{L^p(\mathbb{R}^2)}, \qquad p < \infty$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}^3_+)} \leq C(p)\|g\|_{L^p(\mathbb{R}^2)}, \qquad p = \infty.$$

The following lemma regarding the strong convergence of solutions to the Laplace equation with Neumann boundary data shall be useful in the proof of Theorem 3.1. Of particular importance is the fact that the convergence holds up to the boundary $z = 0$ when $p > \frac{4}{3}$, providing a stronger result than interior regularity estimates.

Lemma 3.2.5. *Let $\{g_\epsilon\}_{\epsilon>0}$ be a bounded sequence of functions in $L^p(\mathbb{R}^2)$ for $p > \frac{4}{3}$. Let $u_\epsilon(z,x) : \mathbb{R}^3_+ \to \mathbb{R}$ be the solution to*

$$\begin{cases} \Delta u_\epsilon = 0 & z > 0 \\ \partial_\nu u_\epsilon = g_\epsilon & z = 0 \end{cases}$$

Then there exists u such that up to a subsequence, ∇u_ϵ converges strongly to ∇u in the spaces $L^2\left((0,R) \times B_R(0)\right)$ for all $R > 0$.

3.3 Proof of Theorem 3.1: The Existence of Weak Solutions

We now have the estimates necessary for the proof of the main theorem. Here we assume that p and q satisfy the assumptions of Theorem 3.1.

Proof of Theorem 3.1. Let $\{\gamma_\epsilon\}_{\epsilon>0}$ be a sequence of approximate identities compactly supported in $B_\epsilon(0)$ in $\mathbb{R}^2$ and $\{\Gamma_\epsilon\}_{\epsilon>0}$ a sequence of approximate identities compactly supported in $B_\epsilon(0)$ in $\mathbb{R}^3$. We define truncated versions of the initial data and forcing by

$$\omega_{T_\epsilon} = \omega \mathcal{X}_{\{|\omega|<\frac{1}{\epsilon}, |(z,x)|<\frac{1}{\epsilon}\}}, \qquad \theta_{T_\epsilon} = \theta \mathcal{X}_{\{|\theta|<\frac{1}{\epsilon}, |x|<\frac{1}{\epsilon}\}},$$

with $f_{L,T_\epsilon}(t)$ and $f_{\nu,T_\epsilon}(t)$ defined analogously for each time $t \geq 0$. Then we regularize by putting

$$\omega_\epsilon = \Gamma_\epsilon * \omega_{T_\epsilon}, \qquad \theta_\epsilon = \gamma_\epsilon * \theta_{T_\epsilon}, \qquad f_{L,\epsilon}(t) = \Gamma_\epsilon * f_{L,T_\epsilon}(t), \qquad f_{\nu,\epsilon}(t) = \gamma_\epsilon * f_{\nu,T_\epsilon}(t),$$

ensuring that ω_ϵ, θ_ϵ, $f_{L,\epsilon}(t)$, and $f_{\nu,\epsilon}(t)$ are compactly supported, C^∞ functions in space for each $t \geq 0$. Setting $\Delta\Psi_{0,\epsilon} := \omega_\epsilon$ and $\partial_\nu \Psi_{0,\epsilon} := \theta_\epsilon$ shows that the assumptions of Theorem 3.2.1 are satisfied. Therefore there exists a classical solution $\nabla\Psi_\epsilon$ to

$$\begin{cases} \partial_t(\Delta\Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\Delta\Psi_\epsilon) = f_{L,\epsilon} & t > 0, \quad z > 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi_\epsilon) + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}(\partial_\nu\Psi_\epsilon) = f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}}(\partial_\nu\Psi_\epsilon) & t > 0, \quad z = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2 \end{cases}$$

Define $F_\epsilon : [0,\infty) \times \mathbb{R}^3_+ \to \mathbb{R}$ for all time by

$$\begin{cases} \Delta F_\epsilon = f_{L,\epsilon} & z > 0 \\ \partial_\nu F_\epsilon = f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}} \partial_\nu \Psi_\epsilon & z = 0 \end{cases}$$

Integrating by parts with a smooth test function $\phi(t,z,x)$ with compact spacial support in $\overline{\mathbb{R}^3_+}$, we have the following equalities:

$$\begin{aligned} -\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \Big(\Big(\partial_t \nabla\phi + \overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla}\nabla\phi \Big) \cdot \nabla\Psi_\epsilon + \nabla\phi \cdot \nabla F_\epsilon \Big)\, dx\, dz\, dt \\ = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \Big(\Big(\partial_t \phi + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}\phi \Big) \Delta\Psi_\epsilon + \phi \Delta F_\epsilon \Big)\, dx\, dz\, dt \\ - \int_0^T \int_{\mathbb{R}^2} \Big(\Big(\partial_t \phi + \overline{\nabla}^\perp \Psi_\epsilon \cdot \overline{\nabla}\phi \Big) \partial_\nu \Psi_\epsilon + \phi \partial_\nu F_\epsilon \Big)\, dx\, dt \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^2} \nabla\phi(0,z,x) \cdot \nabla\Psi_\epsilon(0,z,x)\, dx\, dz = -\int_0^\infty \int_{\mathbb{R}^2} \phi(0,z,x) \Delta\Psi_\epsilon(0,z,x)\, dx\, dz \\ + \int_{\mathbb{R}^2} \phi(0,0,x) \partial_\nu \Psi_\epsilon(0,0,x)\, dx \end{aligned}$$

Using that $\nabla\Psi_\epsilon$ is a solution to the regularized system, the right hand sides of the above equalities are in fact equal, and therefore the left hand sides are equal as well, i.e.

$$\begin{aligned} -\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \Big(\Big(\partial_t \nabla\phi + \overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla}\nabla\phi \Big) \cdot \nabla\Psi_\epsilon + \nabla\phi \cdot \nabla F_\epsilon \Big)\, dx\, dz\, dt \\ = \int_0^\infty \int_{\mathbb{R}^2} \nabla\phi(0,z,x) \cdot \nabla\Psi_\epsilon(0,z,x)\, dx\, dz \end{aligned} \tag{3.3}$$

If the support of ϕ is not compact but $\Delta\phi$ and $\partial_\nu\phi$ are compactly supported, we claim the equality (3.3) still holds under approximation by smooth functions. By Lemma B.0.2, $\nabla^2\phi \in L^{1+\epsilon} \cap L^\infty(\mathbb{R}^3_+)$, ensuring that

$$\overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla}\nabla\phi \cdot \nabla\Psi_\epsilon$$

is bounded using Hölder's inequality and we can pass to the limit from a sequence of compactly supported functions. In addition, $\nabla\phi \in L^2 \cap L^\infty(\mathbb{R}^3_+)$, ensuring that $\nabla\phi \cdot \nabla F_\epsilon$ is well-defined by the assumptions on the integrability of f_ν and f_L.

To pass to the limit in (3.3), we use Theorem 3.2.1 to detail the spaces in which $\{\Psi_\epsilon\}$ is pre-compact. Throughout, $T > 0$ is fixed, and weak-$*$ convergence is abbreviated simply as weak convergence. We decompose $\Psi_\epsilon(t) = \Psi_{\epsilon,1}(t) + \Psi_{\epsilon,2}(t)$ as follows:

$$\begin{cases} \Delta\Psi_{\epsilon,1} = 0 \\ \partial_\nu \Psi_{\epsilon,1} = \partial_\nu \Psi_\epsilon \end{cases} \qquad \begin{cases} \Delta\Psi_{\epsilon,2} = \Delta\Psi_\epsilon \\ \partial_\nu \Psi_{\epsilon,2} = 0. \end{cases}$$

1. By Theorem 3.2.1(2), $\{\partial_\nu \Psi_{\epsilon,1}\}$ is bounded in $L^\infty([0,T]; L^p(\mathbb{R}^2))$ and we can pass to a weakly convergent subsequence.

2. By Theorem 3.2.1(1), $\{\Psi_{\epsilon,2}\}$ is bounded in $L^\infty([0,T]; \mathring{W}^{2,q}(\mathbb{R}^3_+))$ and we can pass to a weakly convergent subsequence.

Given the weak convergence of $\partial_\nu \Psi_\epsilon = \partial_\nu \Psi_{\epsilon,1}$ and $\Delta\Psi_\epsilon = \Delta\Psi_{\epsilon,2}$, we will show that up to a subsequence, $\nabla\Psi_\epsilon = \nabla\Psi_{\epsilon,1} + \nabla\Psi_{\epsilon,2}$ converges strongly in $L^\infty([0,T]; L^2((0,R) \times B_0(R)))$ for any R. To prove this, we use the Aubin-Lions lemma [2] (as do Puel and Vasseur [82]); note also that here is where require $p > \frac{4}{3}$ and $q > \frac{6}{5}$. We break the argument into steps. The first step specifies the Banach space in which $\{\nabla\Psi_\epsilon\}$ is bounded. The second step specifies the Banach space in which $\{\partial_t \nabla\Psi_\epsilon\}$ is bounded. The last step specifies the relationship between these Banach spaces and $L^\infty([0,T]; L^2((0,R) \times B_0(R)))$, justifying the use of the Aubin-Lions lemma.

Step One : Let $\nabla h \in C_c^\infty(\overline{\mathbb{R}^3_+})$. Define

$$\|\nabla h\|_{B_1} := \|\Delta h\|_{L^q(\mathbb{R}^3_+)} + \|\partial_\nu h\|_{L^p(\mathbb{R}^2)}.$$

Define the space of gradients

$$B_1 := \mathrm{cl}(C_c^\infty(\overline{\mathbb{R}^3_+}))$$

to be the closure of $C_c^\infty(\overline{\mathbb{R}^3_+})$ gradients of functions in the upper half space with respect to the norm $\|\cdot\|_{B_1}$. Approximating $\nabla\Psi_\epsilon(t)$ by gradients of smooth, compactly supported functions shows that $\nabla\Psi_\epsilon(t) \in B_1$ for each $t \in [0,T]$, and thus $\{\nabla\Psi_\epsilon\} \subset L^\infty([0,T]; B_1)$ is a bounded sequence.

Step Two : The distributional time derivative $\partial_t \nabla \Psi_\epsilon$ (in the sense of the Aubin-Lions lemma) is defined by the equality

$$\langle \partial_t \nabla \Psi_\epsilon, h \rangle := -\int_0^T \nabla \Psi_\epsilon(t) h'(t)\, dt \tag{3.4}$$

for all $h \in C_c^\infty(0,T)$. Define the Sobolev space V to be the closure of $C_c^\infty([0,R) \times B_0(R))$ vector fields under the usual $H^3(\mathbb{R}^3_+)$ norm, and set B_{-1} to be the dual space V^*. To show that $\{\partial_t \nabla \Psi_\epsilon\}$ is a bounded sequence in $L^\infty([0,T]; B_{-1})$, we test (3.4) against a vector field $v \in V$. By Lemma B.0.2, we have that

$$-\int_0^T \int_{\mathbb{R}^3_+} \nabla \Psi_\epsilon(t) h'(t) v(z,x)\, dt\, dz\, dx = -\int_0^T \int_{\mathbb{R}^3_+} \nabla \Psi_\epsilon(t) h'(t) \nabla w(z,x)\, dt\, dz\, dx$$

Using again Lemma B.0.2, we have that $\nabla w \in H^3(\mathbb{R}^3_+)$ and $\nabla^2 w \in L^{1+\delta}(\mathbb{R}^3_+)$ for any $\delta > 0$. The assumptions on the integrability of f_ν and f_L in Theorem 3.1 ensure that $\nabla F_\epsilon \in L^1_t(L^2(\mathbb{R}^3_+))$, and therefore $\nabla F_\epsilon \cdot \nabla w$ is well-defined and integrable independently of ϵ. The assumptions on the integrability of θ, ω, f_L, and f_ν in Theorem 3.1 and the estimates in Theorem 3.2.1 ensure that $\nabla \Psi_\epsilon(t)$ always belongs to $L^{\frac{3p}{2}} + L^{\frac{3q}{3-q}}(\mathbb{R}^3_+)$ for some $p < \infty$, $q < 3$ uniformly in t and ϵ, and therefore

$$\overline{\nabla}^\perp \Psi_\epsilon : \overline{\nabla} \nabla w \cdot \nabla \Psi_\epsilon$$

is well-defined and integrable uniformly in ϵ by Hölder's inequality. Thus all terms in equality (3.3) are well-defined and bounded uniformly in ϵ for the test function $\phi = h(t) w(z,x)$. In conclusion, we have that $\{\partial_t \nabla \Psi_\epsilon\}$ is a bounded sequence in $L^\infty([0,T]; B_{-1})$.

Step Three : The inclusion of $\left(L^2((0,R) \times B_0(R))\right)^3$ into B_{-1} is continuous. We now show that the inclusion of B_1 into $\left(L^2((0,R) \times B_0(R))\right)^3$ is compact. Given a bounded sequence $\{\nabla z_n\} \subset B_1$, decompose as follows:

$$\begin{cases} \Delta z_{n,1} = 0 \\ \partial_\nu z_{n,1} = \partial_\nu z_n \end{cases} \qquad \begin{cases} \Delta z_{n,2} = \Delta z_n \\ \partial_\nu z_{n,2} = 0. \end{cases}$$

Since $p > \frac{4}{3}$, we can apply Lemma 3.2.5 to $\{\nabla z_{n,1}\}$, yielding strong convergence of a subsequence in $\left(L^2((0,R) \times B_0(R))\right)^3$. Using that $z_{n,2} \in \mathring{W}^{2,q}(\mathbb{R}^3_+)$ and $q > \frac{6}{5}$, the Rellich-Kondrachov theorem yields in addition strong convergence of a subsequence $\{\nabla z_{n,2}\}$ in

$(L^2((0,R)\times B_0(R)))^3$. Summing $z_n = z_{n,1} + z_{n,2}$ gives that ∇z_n converges strongly in $(L^2((0,R)\times B_0(R)))^3$, and therefore B_1 embeds compactly in $(L^2((0,R)\times B_0(R)))^3$. Therefore the Aubin-Lions lemma can be applied, and up to a subsequence,

$$\nabla\Psi_\epsilon \to \nabla\Psi \quad \text{in} \quad L^\infty\left([0,T];\left(L^2((0,R)\times B_0(R))\right)^3\right) \tag{3.5}$$

We then diagonalize the subsequence to obtain strong convergence for any $R\in\mathbb{N}$.

Returning to the proof of Theorem 3.1, let $\nabla\Psi$ be the limit of $\nabla\Psi_\epsilon$ with convergence in the spaces specified in (1)-(2) and (3.5). By (1) and integration by parts,

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\nabla\phi\cdot\nabla F_\epsilon\,dx\,dz\,dt &= \lim_{\epsilon\to 0}\Bigg(-\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\phi f_{L,\epsilon}\,dx\,dz\,dt \\
&\qquad + \int_0^T\int_{\mathbb{R}^2}\phi\left(f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}}\partial_\nu\Psi_\epsilon\right)\,dx\,dt\Bigg) \\
&= \lim_{\epsilon\to 0}\left(-\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\phi f_{L,\epsilon} + \int_0^T\int_{\mathbb{R}^2}\left(\phi f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}}\phi\partial_\nu\Psi_{\epsilon,1}\right)\right) \\
&= -\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\phi f_L + \int_0^T\int_{\mathbb{R}^2}\phi f_\nu \\
&= \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\nabla\phi\cdot\nabla F
\end{aligned}$$

for F solving the boundary value problem $\Delta F = f_L$ and $\partial_\nu F = f_\nu$. Second, by (3.5),

$$\lim_{\epsilon\to 0}\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\overline{\nabla}^\perp\Psi_\epsilon : \overline{\nabla}\nabla\phi\right)\cdot\nabla\Psi_\epsilon\,dx\,dz\,dt = \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\overline{\nabla}^\perp\Psi : \overline{\nabla}\nabla\phi\right)\cdot\nabla\Psi\,dx\,dz\,dt$$

In addition, it is immediate that

$$\lim_{\epsilon\to 0}\int_0^\infty\int_{\mathbb{R}^2}\nabla\phi(0,z,x)\cdot\nabla\Psi_\epsilon(0,z,x)\,dx\,dz = \int_0^\infty\int_{\mathbb{R}^2}\nabla\phi(0,z,x)\cdot\nabla\Psi(0,z,x)\,dx\,dz$$

and

$$\lim_{\epsilon\to 0}\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\partial_t\nabla\phi\cdot\nabla\Psi_\epsilon\,dx\,dz\,dt = \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\partial_t\nabla\phi\cdot\nabla\Psi\,dx\,dz\,dt.$$

Passing to the limit in (3.3), we have that

$$\begin{aligned}
-\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\nabla\phi + \overline{\nabla}^\perp\Psi : \overline{\nabla}\nabla\phi\right)\cdot\nabla\Psi + \nabla\phi\cdot\nabla F\right)\,dx\,dz\,dt \\
= \int_0^\infty\int_{\mathbb{R}^2}\nabla\phi(0,z,x)\cdot\nabla\Psi(0,z,x)\,dx\,dz
\end{aligned}$$

and thus Ψ satisfies Definition 3.1.1. The bound in the statement of the theorem follows from passing to the limit in ϵ in Theorem 3.2.1(1)-(3), completing the proof. □

3.4 Proof of Theorem 3.2: Showing That Different Notions of Weak Solutions Coincide

We divide up the proof into parts (1), (2) and (3).

Proof of Theorem 3.2(1). The first step shows that integration by parts is valid for the reformulated equation, and the second step then integrates by parts to prove the claim.

Step One : First, we extend the Sobolev function Ψ to $\mathbb{R}^3$, denoting the extended function by Ψ_E. Let $\{\Gamma_\epsilon\}_{\epsilon>0}$ be a sequence of approximate identities in $\mathbb{R}^3$. Define

$$\nabla \Psi_{E,\epsilon} := \nabla \Psi_E * \Gamma_\epsilon$$

for $\epsilon > 0$. By assumption, we have

$$\Delta \Psi \in L^\infty\left([0,T); L^q(\mathbb{R}^3_+)\right), \qquad \partial_\nu \Psi \in L^\infty\left([0,T); L^p(\mathbb{R}^2)\right)$$

for $q \in [\frac{3}{2}, 3]$ and $p \in [2,\infty]$. Combined with the elliptic estimates in Lemma 3.2.3 and Lemma 3.2.4, this ensures that integration by parts for $\nabla\Psi_{E,\epsilon}$ is valid, and thus for ϕ compactly supported in $\mathbb{R}^3_+$ and time,

$$\begin{aligned}
-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} &\left(\left(\partial_t \nabla\phi + \overline{\nabla}^\perp \Psi_{E,\epsilon} : \overline{\nabla}\nabla\phi\right) \cdot \nabla\Psi_{E,\epsilon} + \nabla\phi\cdot\nabla F\right) dx\,dz\,dt \\
&= \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon}\cdot\overline{\nabla}\phi\right)\Delta\Psi_{E,\epsilon} + \phi\Delta F\right) dx\,dz\,dt \\
&\quad - \int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon}\cdot\overline{\nabla}\phi\right)\partial_\nu\Psi_{E,\epsilon} + \phi\partial_\nu F\right) dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^2} \nabla\phi(0,z,x)\cdot\nabla\Psi_\epsilon(0,z,x)\,dx\,dz = &-\int_0^\infty\int_{\mathbb{R}^2}\phi(0,z,x)\Delta\Psi_\epsilon(0,z,x)\,dx\,dz \\
&+ \int_{\mathbb{R}^2}\phi(0,0,x)\partial_\nu\Psi_\epsilon(0,0,x)\,dx.
\end{aligned}$$

We now argue that passing to the limit is justified in each identity. We have that Lemma 3.2.3, Lemma 3.2.4 give that $\nabla\Psi \in L^{\frac{3q}{3-q}}(\mathbb{R}^3_+) + L^{\frac{3p}{2}}(\mathbb{R}^3_+)$ for all time. Noticing that since $q \geq \frac{3}{2}$ and $p \geq 2$, we have that

$$\frac{3q}{3-q} \geq 3, \quad \frac{3p}{2} \geq 3,$$

and the following convergences follow:

$$\Delta\Psi_{E,\epsilon} \to \Delta\Psi \quad \text{in} \quad L^2\left([0,T); L^{\frac{3}{2}}_{loc}(\mathbb{R}^3_+)\right)$$

$$\nabla\Psi_{E,\epsilon} \to \nabla\Psi \quad \text{in} \quad L^2\left([0,T); L^3_{loc}(\mathbb{R}^3_+)\right).$$

Furthermore, using Hölder's inequality shows that for each fixed time, $\overline{\nabla}^{\perp}\Psi\Delta\Psi \in L^1_{loc}(\mathbb{R}^3_+)$ and $\overline{\nabla}^{\perp}\Psi \otimes \nabla\Psi \in L^1_{loc}(\mathbb{R}^3_+)$. Therefore,

$$\overline{\nabla}^{\perp}\Psi_{E,\epsilon}\Delta\Psi_{E,\epsilon} \to \overline{\nabla}^{\perp}\Psi\Delta\Psi \quad \text{in} \quad L^1\left([0,T); L^1_{loc}(\mathbb{R}^3_+)\right)$$

$$\overline{\nabla}^{\perp}\Psi_{E,\epsilon} \otimes \nabla\Psi_{E,\epsilon} \to \overline{\nabla}^{\perp}\Psi \otimes \nabla\Psi \quad \text{in} \quad L^1\left([0,T); L^1_{loc}(\mathbb{R}^3_+)\right)$$

Finally, we have that

$$\frac{2q}{3-q} \geq 2,$$

and Lemma A.0.1 gives $\nabla\Psi_2|_{z=0} \in L^{\frac{2q}{3-q}}(\mathbb{R}^2)$. Recalling that $\partial_\nu\Psi \in L^2(\mathbb{R}^2)$ and $\overline{\nabla}^{\perp}\Psi_1 = -\mathcal{R}^{\perp}\partial_\nu\Psi$, applying Hölder again gives $\overline{\nabla}^{\perp}\Psi\partial_\nu\Psi \in L^1_{loc}(\mathbb{R}^2)$. It therefore follows that

$$\partial_\nu\Psi_{E,\epsilon} \to \partial_\nu\Psi \quad \text{in} \quad L^2\left([0,T); L^2_{loc}(\mathbb{R}^2)\right)$$

$$\overline{\nabla}^{\perp}\Psi_{E,\epsilon}|_{z=0} \to \overline{\nabla}^{\perp}\Psi|_{z=0} \quad \text{in} \quad L^2\left([0,T); L^2_{loc}(\mathbb{R}^2)\right)$$

and

$$\overline{\nabla}^{\perp}\Psi_{E,\epsilon}\partial_\nu\Psi_{E,\epsilon} \to \overline{\nabla}^{\perp}\Psi\partial_\nu\Psi \quad \text{in} \quad L^1\left([0,T); L^1_{loc}(\mathbb{R}^2)\right)$$

Letting ϵ tend to 0 shows that

$$\begin{aligned} -\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\nabla\phi + \overline{\nabla}^{\perp}\Psi : \overline{\nabla}\nabla\phi\right)\cdot\nabla\Psi + \nabla\phi\cdot\nabla F\right)dx\,dz\,dt \\ = \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\phi + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\phi\right)\Delta\Psi + \phi\Delta F\right)dx\,dz\,dt \\ - \int_0^T\int_{\mathbb{R}^2}\left(\left(\partial_t\phi + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\phi\right)\partial_\nu\Psi + \phi\partial_\nu F\right)dx\,dt. \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \int_0^\infty\int_{\mathbb{R}^2}\nabla\phi(0,z,x)\cdot\nabla\Psi(0,z,x)\,dx\,dz = -\int_0^\infty\int_{\mathbb{R}^2}\phi(0,z,x)\Delta\Psi(0,z,x)\,dx\,dz \\ + \int_{\mathbb{R}^2}\phi(0,0,x)\partial_\nu\Psi(0,0,x)\,dx. \end{aligned} \tag{3.7}$$

Step Two : Let us start by assuming that $\nabla\Psi$ satisfies Definition 3.1.1. Then we have that

$$-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla\phi + \overline{\nabla}^\perp \Psi : \overline{\nabla}\nabla\phi \right) \cdot \nabla\Psi + \nabla\phi \cdot \nabla F \right) \, dx \, dz \, dt = \int_0^\infty \int_{\mathbb{R}^2} \nabla\phi(0,z,x) \cdot \nabla\Psi(0,z,x) \, dx \, dz,$$

i.e. the left hand side of (3.6) is equal to the left hand side of (3.7). Choosing ϕ to be compactly supported in $[-T,T] \times \mathbb{R}^3_+$ gives that

$$\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\phi \right) \cdot \Delta\Psi + \phi \Delta F \right) \, dx \, dz \, dt$$
$$= -\int_0^\infty \int_{\mathbb{R}^2} \phi(0,z,x) \Delta\Psi(0,z,x) \, dx \, dz,$$

and therefore $\nabla\Psi$ satisfies (3.1).

To show that $\nabla\Psi$ satisfies (3.2), choose $\bar{\phi}$ to be a test function compactly supported in $[-T,T] \times \mathbb{R}^2$. Let $\gamma(z)$ be a smooth function of one variable compactly supported in $[-1,1]$ with $\gamma \equiv 1$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$. Let $\gamma_n(z) = \gamma(nz)$. Define $\phi_n(t,z,x) = \gamma_n(z)\bar{\phi}(t,x)$. Then $\overline{\nabla}\phi_n$, $\partial_t \phi_n$, and ϕ_n converge to 0 in $\mathbb{R}^3_+$ (both pointwise and in any Lebesgue space). We have that the right hand side of (3.6) is equal to the right hand side of (3.7). Then plugging in ϕ_n as a test function, letting n tend to infinity, and passing to the limit shows that

$$-\int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t \bar{\phi} + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\bar{\phi} \right) \partial_\nu \Psi + \bar{\phi} \partial_\nu F \right) \, dx \, dt$$
$$= \int_{\mathbb{R}^2} \bar{\phi}(0,0,x) \partial_\nu \Psi(0,x) \, dx. \tag{3.8}$$

Now assume for the other direction that Ψ verifies Definition 3.1.2. Then for ϕ compactly supported in $\mathbb{R}^3_+$ (and time) and $\bar{\phi}$ compactly supported in $\mathbb{R}^2$ (and time),

$$-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}\phi \right) \Delta\Psi + \phi f_L \right) \, dx \, dz \, dt$$
$$= \int_0^\infty \int_{\mathbb{R}^2} \phi(0,z,x) \Delta\Psi(0,z,x) \, dx \, dz \tag{3.9}$$

and

$$-\int_0^T \int_{\mathbb{R}^2} \left(\left(\partial_t \bar{\phi} + \overline{\nabla}^\perp \Psi(t,0,x) \cdot \overline{\nabla}\bar{\phi} \right) \partial_\nu \Psi(t,x) + \bar{\phi} f_\nu \right) \, dx \, dt$$
$$= \int_{\mathbb{R}^2} \bar{\phi}(0,x) \partial_\nu \Psi(0,x) \, dx.$$

Before proceeding we show that (3.9) holds for ϕ compactly supported in $\mathbb{R}^3$ rather than $\mathbb{R}^3_+$. Let ϕ be compactly supported in $\mathbb{R}^3$ and time. Using $\gamma_n(z)$ as defined previously, define

$$\phi_n(t,z,x) = (1-\gamma_n(z))\,\phi(t,z,x).$$

Then ϕ_n is compactly supported in $\mathbb{R}^3_+$ and $\overline{\nabla}\phi_n$, $\partial_t\phi_n$, and ϕ_n converge to $\overline{\nabla}\phi$, $\partial_t\phi$, and ϕ respectively, both pointwise in $\mathbb{R}^3_+$ and in any Lebesgue space. Therefore

$$\begin{aligned}
-\int_0^T\int_0^\infty\int_{\mathbb{R}^2}&\left(\left(\partial_t\phi+\overline{\nabla}^\perp\Psi\cdot\overline{\nabla}\phi\right)\Delta\Psi+\phi f_L\right)\,dx\,dz\,dt\\
&=\lim_{n\to\infty}-\int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\phi_n+\overline{\nabla}^\perp\Psi\cdot\overline{\nabla}\phi_n\right)\Delta\Psi+\phi_n f_L\right)\,dx\,dz\,dt\\
&=\lim_{n\to\infty}\int_0^\infty\int_{\mathbb{R}^2}\phi_n(0,z,x)\Delta\Psi(0,z,x)\,dx\,dz\\
&=\int_0^\infty\int_{\mathbb{R}^2}\phi(0,z,x)\Delta\Psi(0,z,x)\,dx\,dz,.
\end{aligned}$$

We have then that the right hand side of (3.6) is equal to the right hand side of (3.7), showing then that the left hand side of (3.6) is equal to the left hand side of (3.7). Therefore, $\nabla\Psi$ satisfies Definition 3.1.1 and is a weak solution to (rQG).

□

Proof of Theorem 3.2(2). As in part (1), the proof is split up into two steps.

Step One : We assume that $p\in(\frac{4}{3},2]$, $q\in[\frac{3}{2},3]$, and

$$p\ge\frac{2q}{3(q-1)}.$$

Let us first point out the implications of the assumptions on p and q. Throughout, we use the definitions of Ψ_1 and Ψ_2 described in Definition 3.1.3. First, since $q\ge\frac{3}{2}$, Lemma 3.2.3 ensures that for all time, $\nabla\Psi_2\in L^3_{loc}(\mathbb{R}^3_+)$, and therefore $\nabla\Psi_2\Delta\Psi\in L^1_{loc}(\mathbb{R}^3_+)$ is well-defined by Hölder's inequality. Secondly, from Lemma 3.2.4, we have $\nabla\Psi_1\in L^{\frac{3p}{2}}(\mathbb{R}^3_+)$. Thus, the assumption that $p\ge\frac{2q}{3(q-1)}$ ensures that

$$\frac{2}{3p}\le\frac{q-1}{q},$$

and therefore $\nabla\Psi_1\Delta\Psi \in L^1_{loc}(\mathbb{R}^3_+)$ is also well-defined by Hölder's inequality. Next, applying Lemma A.0.1 to Ψ_2 gives that $\nabla\Psi_2|_{z=0} \in L^{\frac{2q}{3-q}}(\mathbb{R}^2)$. Using that $p \geq \frac{2q}{3(q-1)}$, it follows that

$$\frac{3-q}{2q} \leq \frac{p-1}{p}.$$

Therefore, $\overline{\nabla}^\perp\Psi_2|_{z=0}\partial_\nu\Psi \in L^1_{loc}(\mathbb{R}^2)$ is also well-defined from Hölder's inequality. Combined with the fact that $p > \frac{4}{3}$, we can apply Lemma 3.2.2, yielding that

$$\left(\overline{\nabla}^\perp\Psi\partial_\nu\Psi\right)_C$$

is well-defined as a distribution.

The proof now proceeds as before. We regularize and extend $\nabla\Psi$ to $\nabla\Psi_{E,\epsilon}$. Then

$$\begin{aligned}
-\int_0^T\int_0^\infty\int_{\mathbb{R}^2} &\left(\left(\partial_t\nabla\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon} : \overline{\nabla}\nabla\phi\right)\cdot\nabla\Psi_{E,\epsilon} + \nabla\phi\cdot\nabla F\right)\,dx\,dz\,dt \\
&= \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon}\cdot\overline{\nabla}\phi\right)\Delta\Psi_{E,\epsilon} + \phi\Delta F\right)\,dx\,dz\,dt \\
&\quad - \int_0^T\int_{\mathbb{R}^2}\left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon}\cdot\overline{\nabla}\phi\right)\partial_\nu\Psi_{E,\epsilon} + \phi\partial_\nu F\right)\,dx\,dt \\
&= \int_0^T\int_0^\infty\int_{\mathbb{R}^2}\left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi_{E,\epsilon}\cdot\overline{\nabla}\phi\right)\Delta\Psi_{E,\epsilon} + \phi\Delta F\right)\,dx\,dz\,dt \\
&\quad - \int_0^T\int_{\mathbb{R}^2}\left(\partial_t\phi\partial_\nu\Psi_{E,\epsilon} + \left(\overline{\nabla}^\perp\Psi_{E,\epsilon}\partial_\nu\Psi_{E,\epsilon}\right)_C\cdot\overline{\nabla}\phi + \phi\partial_\nu F\right)\,dx\,dt.
\end{aligned}$$

The second equality holds since the smoothness of $\nabla\Psi_{E,\epsilon}$ ensures that $\left(\overline{\nabla}^\perp\Psi_{E,\epsilon}\partial_\nu\Psi_{E,\epsilon}\right)_C$ as a distribution is equal to the regular distribution $\left(\overline{\nabla}^\perp\Psi_{E,\epsilon}\partial_\nu\Psi_{E,\epsilon}\right)$ (see Lemma 3.2.2). We have

$$\begin{aligned}
\Delta\Psi_{E,\epsilon} \to \Delta\Psi \qquad &\text{in} \qquad L^2\left([0,T];L^q(\mathbb{R}^3_+)\right) \\
\nabla\Psi_{E,\epsilon} \to \nabla\Psi \qquad &\text{in} \qquad L^2\left([0,T];L^{\frac{q}{q-1}}_{loc}(\mathbb{R}^3_+)\right),
\end{aligned}$$

and therefore

$$\nabla\Psi_{E,\epsilon}\Delta\Psi_{E,\epsilon} \to \nabla\Psi\Delta\Psi \qquad \text{in} \qquad L^1\left([0,T];L^1_{loc}(\mathbb{R}^3_+)\right)$$

and

$$\overline{\nabla}^\perp\Psi_{E,\epsilon}\otimes\nabla\Psi_{E,\epsilon} \to \overline{\nabla}^\perp\Psi\nabla\Psi \qquad \text{in} \qquad L^1\left([0,T];L^1_{loc}(\mathbb{R}^3_+)\right)$$

We have

$$\partial_\nu \Psi_{E,\epsilon} \to \partial_\nu \Psi \qquad \text{in} \qquad L^2\left([0,T); L^p(\mathbb{R}^2)\right)$$

and the weak-* convergence

$$\partial_\nu \Psi_{E,\epsilon} \to \partial_\nu \Psi \qquad \text{in} \qquad L^\infty\left([0,T); L^p(\mathbb{R}^2)\right).$$

In addition,

$$\overline{\nabla}^\perp \Psi_{E,\epsilon,2}|_{z=0} \to \overline{\nabla}^\perp \Psi_2|_{z=0} \qquad \text{in} \qquad L^2\left([0,T); L^{\frac{p}{p-1}}_{loc}(\mathbb{R}^2)\right),$$

and applying Lemma 3.2.2 since $p > \frac{4}{3}$ yields that

$$\begin{aligned}\overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi_{E,\epsilon} \partial_\nu \Psi_{E,\epsilon}\right)_C &= \overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi_{E,\epsilon,2} \partial_\nu \Psi_{E,\epsilon} - \partial_\nu \Psi_{E,\epsilon} \mathcal{R}^\perp \partial_\nu \Psi_{E,\epsilon}\right) \\ &\to \overline{\nabla} \cdot \left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi\right)_C \qquad \text{in} \qquad \mathcal{D}'\left((0,T) \times \mathbb{R}^2\right).\end{aligned}$$

Passing to the limit shows that

$$\begin{aligned}&-\int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \nabla \phi + \overline{\nabla}^\perp \Psi : \overline{\nabla} \nabla \phi\right) \cdot \nabla \Psi + \nabla \phi \cdot \nabla F\right) dx\, dz\, dt \\ &\qquad = \int_0^T \int_0^\infty \int_{\mathbb{R}^2} \left(\left(\partial_t \phi + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla} \phi\right) \cdot \Delta \Psi + \phi \Delta F\right) dx\, dz\, dt \\ &\qquad\quad - \int_0^T \int_{\mathbb{R}^2} \left(\partial_t \phi \partial_\nu \Psi + \left(\overline{\nabla}^\perp \Psi \partial_\nu \Psi\right)_C \cdot \overline{\nabla} \phi + \phi \partial_\nu F\right) dx\, dt \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^2} \nabla \phi(0,z,x) \cdot \nabla \Psi(0,z,x)\, dx\, dz = &-\int_0^\infty \int_{\mathbb{R}^2} \phi(0,z,x) \Delta \Psi(0,z,x)\, dx\, dz \\ &+ \int_{\mathbb{R}^2} \phi(0,0,x) \partial_\nu \Psi(0,0,x)\, dx,. \end{aligned} \tag{3.11}$$

Step Two : Assuming (3.10) and (3.11) hold, we can argue precisely as in the proof of Theorem 3.2(1) to prove the theorem. We refer the reader to the proof of Theorem 3.2(1) for further details. □

Proof of Theorem 3.2(3). The claim follows immediately from the observation that since Ψ_1 is harmonic,

$$\overline{\nabla}^\perp \Psi_1 = (0, -\partial_{x_2} \Psi_1, \partial_{x_1} \Psi_1) = (0, \mathcal{R}_2(\partial_\nu \Psi_1), -\mathcal{R}_1(\partial_\nu \Psi_1)) = -\mathcal{R}^\perp(\partial_\nu \Psi_1)$$

and the claim in Lemma 3.2.2 that

$$\int_{\mathbb{R}^2} f\mathcal{R}^{\perp} f \cdot \overline{\nabla}\phi = -\frac{1}{2}\int_{\mathbb{R}^2} \left(\mathcal{R}^{\perp} f\right) \cdot \left([\bar{\Lambda}, \overline{\nabla}\phi]\left(\bar{\Lambda}^{-1} f\right)\right)$$

for $f \in L^2(\mathbb{R}^2)$. $\square$

Chapter 4

Existence of Weak Solutions and Lateral Boundary Conditions for the Inviscid Model in a Cylinder

4.1 Overview

In this chapter we study the inviscid three-dimensional quasi-geostrophic system posed on a bounded domain. The model we consider includes two coupled transport equations as follows:

$$\begin{cases} \left(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\right)(\mathcal{L}(\Psi) + \beta_0 y) = a_L & \Omega \times [0,h] \times [0,T] \\ \left(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\right)(\partial_\nu \Psi) = a_\nu & \Omega \times \{0,h\} \times [0,T]. \end{cases} \tag{QG}$$

In Chapters 2 and 3, the model is posed for $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{T}^2$ and $h = \infty$. However, throughout the remainder of the chapter, the system shall be posed on a fixed cylindrical domain

$$\Omega \times [0,h]$$

where $\Omega \subset \mathbb{R}^2$ is a smooth, bounded set, and the height h is fixed and finite. The functions a_L and a_ν are forcing terms, and β_0 is a parameter coming from the usual β-plane approximation. The normal derivative of Ψ on $\Omega \times \{0,h\}$ is again denoted by $\partial_\nu \Psi$. The operator $\mathcal{L}$ is defined by

$$\mathcal{L} := \partial_{xx} + \partial_{yy} + \partial_z(\lambda \partial_z)$$

where $\lambda > 0$ is a smooth function depending only on z and is related to the density of the fluid. To ensure ellipticity of $\mathcal{L}$ we require

$$\frac{1}{\Lambda} \leq \lambda(z) \leq \Lambda$$

for some $\Lambda \in (0,\infty)$. The values of $\mathcal{L}(\Psi)$ and $\partial_\nu \Psi$ are advected by the fluid velocity field $\overline{\nabla}^{\perp}\Psi$. In order to reconstruct Ψ at each time, it is necessary to supplement the system with a boundary condition on the lateral boundary $\partial\Omega \times [0,h]$.

4.1.1 Boundary Conditions

The purpose of this chapter is to formally derive an appropriate model from the primitive equations while assuming that the lateral boundary is *impermeable*; that is, we assume only that the fluid velocity $\overline{\nabla}^{\perp}\Psi$ is tangent to $\partial\Omega \times [0, h]$. We then prove that weak solutions exist globally in time for the resulting system. In fact, we show in Section 4.2.3 that the impermeability produces two constraints on a possible solution. First, we must have that

$$\Psi(t, x, y, z)|_{\partial\Omega\times[0,h]} = c(t, z) \tag{4.1}$$

for some unknown function $c(t, z)$. However, this is not enough to define a unique solution to an elliptic problem on $\Omega \times [0, h]$. Crucially, the impermeability condition provides another natural constraint. After defining ν_s to be the normal derivative to $\partial\Omega \times \{z\}$ and $d\omega$ the Hausdorff measure on $\partial\Omega$, the second constraint is that for all $z \in [0, h]$,

$$\frac{\partial}{\partial t} \int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi \cdot \nu_s \, d\omega = 0. \tag{4.2}$$

In other words, building a weak solution to (QG) requires choosing a datum $j_0(z) : [0, h] \to \mathbb{R}$ such that for all time,

$$\int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi(t) \cdot \nu_s \, d\omega = j_0(z).$$

These two conditions differentiate the model we derive from closely related models which have been studied recently by Constantin and Nguyen [35], [36] and Constantin and Ignatova [33], [32]. While we shall explain this distinction in detail in Section 4.1.3, we first describe a rough sketch of our existence proof, and then state our main results.

In [82] and Chapter 3, one uses that the transport equations for $\mathcal{L}(\Psi)$ and $\partial_\nu\Psi$ in (QG) preserve the norms of the data for an elliptic problem with Neumann boundary condition. Therefore, a sequence of approximate solutions Ψ_n for which $\mathcal{L}(\Psi_n)$ and $\partial_\nu\Psi_n$ converge weakly in (respectively) $L^\infty_t(L^2(\Omega\times[0, h]))$ and $L^\infty_t(L^2(\Omega\times\{0, h\}))$ will have strong convergence for $\nabla\Psi_n$ in $L^\infty_t(L^2(\Omega \times [0, h]))$. In the setting of the bounded domain $\Omega \times [0, h]$, it is not immediate that imposing (4.1) and (4.2) on the lateral boundary will allow for compactness at the level of $\nabla\Psi_n$ in $L^\infty_t(L^2(\Omega \times [0, h]))$. Indeed, it might seem possible that because (4.2) only controls the average of $\overline{\nabla}\Psi \cdot \nu_s$ on the sides, $\overline{\nabla}\Psi \cdot \nu_s$ could oscillate quite badly on $\partial\Omega \times [0, h]$. To address this, we must formulate (4.2) weakly (see Definition 4.3.1 in Section 3). However, we also prove an elliptic regularity theorem (Theorem 4.3.2) which implies that in fact $\overline{\nabla}\Psi \cdot \nu_s \in L^2(\partial\Omega \times [0, h])$ is well-defined *pointwise*, and $\nabla\Psi_n$ converges strongly to $\nabla\Psi$ in $L^\infty_t(L^2(\Omega \times [0, h]))$. To our knowledge, this type of boundary condition and the corresponding elliptic regularity theorem are novel.

4.1.2 Main Result

Before stating the existence theorem, we must provide several definitions. The first is a natural compatibility condition between the elliptic operator and boundary conditions.

Definition 4.1.1. *Any triple (f, g, j) of functions with $f(x, y, z) \in L^2(\Omega\times[0, h])$, $g(x, y, z) \in L^2(\Omega \times \{0, h\})$, $j(z) \in L^2(0, h)$ is compatible if*

$$\int_{\Omega\times[0,h]} f(x, y, z)\, dx\, dy\, dz = \int_0^h j(z)\, dz + \int_{\Omega\times\{0,h\}} \lambda(z)g(x, y, z)\, dx\, dy.$$

A pair (a_L, a_ν) of forcing terms is compatible if $a_L \in L^1([0, T]; L^2(\Omega \times [0, h]))$, and $a_\nu \in L^1([0, T]; L^2(\Omega \times \{0, h\}))$ for all $T > 0$ with

$$\int_{\Omega\times[0,h]} a_L(x, y, z)\, dx\, dy\, dz = \int_{\Omega\times\{0,h\}} \lambda(z)a_\nu(x, y, z)\, dx\, dy$$

Next, we define the notion of weak solutions to the transport equations in (QG).

Definition 4.1.2. *Let $T > 0$ be given and $\Psi(t, x, y, z) : [0, T] \times \Omega \times [0, h] \to \mathbb{R}$ be such that $\nabla\Psi, \mathcal{L}(\Psi) \in L^\infty([0, T]; L^2(\Omega \times [0, h]))$, $\partial_\nu\Psi \in L^\infty([0, T]; L^2(\Omega \times \{0, h\}))$. Then Ψ is a weak solution to the transport equations in (QG) on $[0, T]$ with initial data f_0 and g_0 and forcing a_L, a_ν if for all $\tilde{\Omega}$ compactly contained in Ω and smooth test functions $\phi(t, x, y, z)$ compactly supported in $[-1, T + 1] \times \tilde{\Omega} \times [-1, h + 1]$*

$$-\int_0^T \int_{\tilde{\Omega}\times[0,h]} \left(\left(\partial_t\phi + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}\phi\right)(\mathcal{L}(\Psi) + \beta_0 y) + \phi a_L\right) dx\, dy\, dz\, dt$$
$$= \int_{\tilde{\Omega}\times[0,h]} \phi|_{t=0} f\, dx\, dy\, dz$$

and

$$\int_0^T \int_{\tilde{\Omega}\times\{0,h\}} \left(\left(\partial_t \phi + \overline{\nabla}^{\perp} \Psi \cdot \overline{\nabla} \phi \right) \partial_\nu \Psi + \phi a_\nu \right) \, dx \, dy \, dt = - \int_{\tilde{\Omega}\times\{0,h\}} \phi|_{t=0} g \, dx \, dy$$

We can now state our existence result.

Theorem 4.1.1. *Let (f_0, g_0, j_0) and (a_L, a_ν) satisfy Definition 4.1.1. Then there exists a global weak solution Ψ to (QG) such that*

1. *$\mathcal{L}(\Psi)|_{t=0} = f_0$, $\partial_\nu \Psi|_{t=0} = g_0$ and Ψ satisfies Definition 4.1.2 for any $T > 0$*

2. *There exists $c(t, z)$ such that for almost every time $t > 0$, $\Psi(t)|_{\partial\Omega\times[0,h]} = c(t, z)$*

3. *For all $t > 0$, $\overline{\nabla}\Psi(t) \cdot \nu_s \in L^2(\partial\Omega \times [0, h])$. If $j_0 \in H^{\frac{1}{2}}(0, h)$, then*
$$\int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi(t) \cdot \nu_s \, d\omega = j_0(z),$$
with the equality holding pointwise in z.

4. *For all time t, $\big(\mathcal{L}(\Psi)(t), \partial_\nu \Psi(t), \overline{\nabla}\Psi \cdot \nu_s(t)\big)$ satisfies the compatibility condition in Definition 4.1.1*

5. *For all $T > 0$ and $t \in [0, T]$, Ψ satisfies the bound*
$$\begin{aligned} &\|\mathcal{L}(\Psi)(t)\|_{L^2(\Omega\times[0,h])} + \|\partial_\nu \Psi(t)\|_{L^2(\Omega\times\{0,h\})} + \|\nabla\Psi(t)\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \\ &\qquad \leq C(\Omega, h, \lambda) \left(\|f\|_{L^2} + \|g\|_{L^2} + \|j\|_{L^2} + \|a_L\|_{L^1([0,T];L^2)} + \|a_\nu\|_{L^1([0,T];L^2)} \right). \end{aligned}$$

4.1.3 Relation to Other Models

Study of the closely related surface quasi-geostrophic equation was initiated by Constantin, Majda, and Tabak [27]. To obtain SQG from (QG), recall that we simplify the model by assuming that $\lambda(z) \equiv 1$, $\beta_0 = 0$, $a_L \equiv a_\nu \equiv 0$, and

$$\Delta\Psi|_{t=0} = 0.$$

As a result, $\Delta\Psi(t) \equiv 0$ uniformly in time, and the entire dynamic is encoded in the equation for $\theta = -\partial_z \Psi|_{z=0} = (-\overline{\Delta})^{\frac{1}{2}}\Psi$

$$\partial_t \theta + \mathcal{R}^{\perp}\theta \cdot \overline{\nabla}\theta = 0. \tag{4.3}$$

Resnick proved global existence of weak solutions for initial data in $L^2(\mathbb{T}^2)$ [83]. Marchand extended Resnick's result to initial data belonging to $L^p(\mathbb{R}^2)$ or $L^p(\mathbb{T}^2)$ for $p > \frac{4}{3}$ [71]. Both the proofs of Resnick and Marchand are based on a reformulation of the nonlinear term using a Cálderon commutator as discussed in Chapter 3.

The techniques used to produce weak solutions by Resnick and Marchand were adapted to bounded domains in a series of papers. In these works the Riesz transform on a bounded domain Ω is defined spectrally using eigenfunctions of the homogenous Dirichlet laplacian. First, Constantin and Ignatova [33], [32] proved nonlinear bounds and commutator estimates for the fractional laplacian and showed the existence of global weak solutions as well as derived interior regularity estimates for (4.3) with added critical dissipation in bounded domains. Constantin and Nguyen [35], [36] then showed the existence of global weak solutions of (4.3) in bounded domains as well as local and global strong solutions for supercritical and critical/subcritical versions of (4.3), respectively.

The weak solutions we construct *cannot coincide* in general with solutions to (4.3) constructed using the spectral Riesz transform. The difference lies in the boundary conditions (4.1) and (4.2). At each time t, we reconstruct Ψ by solving the elliptic problem

$$\begin{cases} \mathcal{L}(\Psi) = f & \Omega \times [0,h] \\ \partial_\nu \Psi = g & \Omega \times \{0,h\} \\ \Psi(x,y,z) = c(z) & \partial\Omega \times \{z > 0\} \\ \int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi \cdot \nu_s = j_0(z) & [0,h]. \end{cases}$$

In particular, we do not require that the stream function Ψ vanishes uniformly on the lateral boundary. While we consider the case of finite height h, the boundary conditions we impose would apply in the case of infinite height as well, which is the most common setting for SQG.

Conversely, let $\{e_n\}$ be the orthonormal basis of eigenfunctions with corresponding eigenvalues $\{\lambda_n\}$ for the homogenous Dirichlet laplacian $-\overline{\Delta}_\Omega$ on Ω, and let

$$\theta = \sum_n a_n(t) e_n(x,y)$$

be a solution to (4.3) posed on the bounded domain Ω. Then the stream function $\Psi|_{z=0}$ is given by

$$\Psi|_{z=0} = \left(-\overline{\Delta}_\Omega\right)^{-\frac{1}{2}} \theta = \sum_n a_n(t) \lambda_n^{-\frac{1}{2}} e_n(x,y),$$

and the harmonic extension for $z \in [0, \infty)$ is given by

$$\Psi(t, x, y, z) = \sum_n a_n(t) e^{-z\sqrt{\lambda_n}} \lambda_n^{-\frac{1}{2}} e_n(x, y).$$

With this definition, Ψ vanishes uniformly on $\partial\Omega \times [0, \infty)$. In addition, if one were to impose (4.2) on a solution to (4.3), then integrating by parts in (x, y) and passing the integral inside the sum gives

$$\sum_n a_n'(t) e^{-z\sqrt{\lambda_n}} \lambda_n^{\frac{1}{2}} \left(\int_\Omega e_n(x, y)\, dx\, dy \right) = 0$$

for all $z > 0$. One can see that this is only satisfied if

$$a_n'(t) \left(\int_\Omega e_n(x, y)\, dx\, dy \right) = 0$$

for all n and $t > 0$, which cannot hold for any bounded domain Ω and initial data. The outline of this paper is as follows; in Section 4.2, we recall the derivation of the system from primitive equations while accounting for the impermeability. In Section 4.3, we produce a solution to the stationary elliptic problem associated to the operator $\mathcal{L}$ and prove an elliptic regularity theorem for the solution. Finally, in Section 4.4, we construct global weak solutions to (QG).

4.2 Derivation from Primitive Equations

4.2.1 Primitive Equations and Re-Scalings

We begin from the so-called primitive equations following the derivation of Bourgeois and Beale [7]. These equations represent the geostrophic balance, which is the balance of the pressure gradient with the Coriolis force. The Boussinesq approximation has been made; that is, changes in density are ignored except when amplified by the effect of gravity. After a re-scaling of the equations, a parameter which varies inversely with the speed of the rotation of the earth called the Rossby number shall appear. Then performing a perturbation expansion in the Rossby number ϵ will yield the stratified system and boundary conditions (4.1) and (4.2). Given a smooth, bounded set $\Omega \subset \mathbb{R}^2$ and a fixed height h, the following equations (after rescaling) will be posed on the cylindrical domain

$$\Omega \times [0, h].$$

We use the notation $\frac{D}{Dt} = \partial_t + \vec{u} \cdot \nabla$ for the material derivative, and the Coriolis force $\mathcal{C} = 2\Theta \sin(\theta)$, where Θ is the angular velocity of the Earth and θ is the latitude. Here (u, v, w) is the fluid velocity, p is the pressure and ρ is the variation in density from a known background density profile $\bar{\varrho}(z)$. That is, the density ϱ satisfies

$$\varrho = \bar{\varrho}(z) + \rho(x, y, z, t).$$

We further assume that the density is decreasing in z and that $-\rho_z$ is bounded above and below away from zero. Throughout, we assume throughout that the fluid velocity is tangent to the boundary.

The primitive equations then are

$$\begin{cases} \frac{Du}{Dt} - \mathcal{C}v = -p_x \\ \frac{Dv}{Dt} + \mathcal{C}u = -p_y \\ \frac{Dw}{Dt} + \rho g = -p_z \\ \nabla \cdot u = 0 \\ \frac{D\varrho}{Dt} = 0. \end{cases}$$

We rescale the equations in such a way so as to remove solutions which vary on a fast time scale. Therefore, we set

$$t = \frac{L}{U} t', \qquad u = Uu', \qquad (x, y, z) = L(x', y', z').$$

Letting θ_0 be a central latitude, we estimate $\mathcal{C}$ using the linear β-plane approximation by

$$\mathcal{C} = 2\Theta \sin(\theta_0) + 2\Theta \cos(\theta_0)(\theta - \theta_0) := \mathcal{C}_0 + 2\Theta \cos(\theta_0)(\theta - \theta_0).$$

The Rossby number ϵ is equal to $\frac{U}{\mathcal{C}_0 L}$. Set $\beta_0 = \frac{\cot(\theta_0)}{\epsilon} \frac{L}{r_0}$. We then have that

$$\begin{aligned} \mathcal{C} &= 2\Theta \sin(\theta_0) + 2\Theta \cos(\theta_0)(\theta - \theta_0) \\ &= \mathcal{C}_0(1 + \epsilon \beta_0 y'). \end{aligned}$$

We assume that $\frac{L}{r_0}$ is $O(\epsilon)$, allowing us to keep the factor of ϵ in front of β_0 even as $\epsilon \to 0$. We scale the density variation by

$$\rho = \frac{\mathcal{C}_0 U}{g} \rho' = \frac{U^2}{\epsilon L g} \rho'$$

and the reference density by

$$\bar{\varrho} = \frac{U^2}{\epsilon^2 L g}\bar{\varrho}'$$

This allows us to write the density non-dimensionally as

$$\varrho = \frac{U^2}{\epsilon^2 L g}(\bar{\varrho}'(z) + \epsilon\rho')$$

Finally, we scale the pressure by $p = \mathcal{C}_0 U L p'$. Applying the scalings to the primitive equations, we obtain

$$\begin{cases} \frac{Du'}{Dt'} - \frac{1}{\epsilon}(1 + \epsilon\beta_0 y')v' = -\frac{1}{\epsilon}p'_{x'} \\ \frac{Dv'}{Dt'} + \frac{1}{\epsilon}(1 + \epsilon\beta_0 y')u' = -\frac{1}{\epsilon}p'_{y'} \\ \frac{Dw'}{Dt'} + \frac{1}{\epsilon}\rho' = -\frac{1}{\epsilon}p'_{z'} \\ \nabla \cdot u' = 0 \\ \frac{D\rho'}{Dt'} + \frac{1}{\epsilon}w'\bar{\varrho}'_{z'} = 0. \end{cases}$$

Let us abuse notation and drop the primes on our scaled equations. Assume that the expansions

$$\vec{u} = \vec{u}(\epsilon) = \vec{u}^{(0)} + \epsilon\vec{u}^{(1)} + O(\epsilon^2)$$

and

$$\rho = \rho(\epsilon) = \rho^{(0)} + \epsilon\rho^{(1)} + O(\epsilon^2)$$

hold. Plugging this ansatz in, we obtain the zero-order equations

$$v^{(0)} = p_x^{(0)}, \qquad u^{(0)} = -p_y^{(0)}, \qquad \rho^{(0)} = -p_z^{(0)}, \qquad w^{(0)} = 0.$$

The last equation follows from the first two equations, the incompressibility (which gives that $w_z^{(0)} = 0$), and the assumption that $w^{(0)} \equiv 0$ on the top and bottom of the domain.

We move now to the first order equations. Let us introduce the notation

$$d_g = \partial_t - p_y^{(0)}\frac{\partial}{\partial_x} + p_x^{(0)}\frac{\partial}{\partial_y}$$

for the zero order geostrophic material derivative. The first order equations are then

$$\begin{cases} d_g(-p_y^{(0)}) - v^{(1)} - \beta_0 y p_x^{(0)} = -p_x^{(1)} \\ d_g(p_x^{(0)}) + u^{(1)} - \beta_0 y p_y^{(0)} = -p_y^{(1)} \\ \rho^{(1)} = -p_z^{(1)} \\ \nabla \cdot u^{(1)} = 0 \\ d_g(-p_z^{(0)}) + w^{(1)}\varrho_z = 0. \end{cases}$$

Let us divide the last equation by $-\frac{1}{\varrho_z}$. We introduce the notation

$$\tilde{\nabla} = (\partial_x, \partial_y, -\frac{1}{\varrho_z}\partial_z).$$

Then we can consolidate the first order equations as

$$\begin{aligned} d_g(\tilde{\nabla} p^{(0)}) + \beta_0(p^{(0)}, 0, 0)^t = (-p_y^{(1)}, p_x^{(1)}, 0)^t - (u^{(1)}, v^{(1)}, w^{(1)})^t \\ - \beta_0 y(-p_y^{(0)}, p_x^{(0)}, 0)^t + \beta_0(p^{(0)}, 0, 0)^t. \end{aligned} \tag{4.4}$$

Note that the right-hand side is divergence free and has no vertical component on the top and bottom boundaries of the domain.

4.2.2 Transporting $\mathcal{L}(\Psi)$ and $\partial_\nu \Psi$

We now take the divergence of (4.4) in order to arrive at (QG). As noted, the divergence of the right hand side is zero. The divergence of $\beta_0(p^{(0)}, 0, 0)^t$ is $\beta_0 p_x^{(0)}$. Examining the transport term $d_g(\tilde{\nabla} p^{(0)})$ and calculating ∂_z of the third component, we obtain

$$d_g(\partial_z(e_3 \cdot \tilde{\nabla} p^{(0)})) + \partial_z u^{(0)} \partial_x(e_3 \cdot \tilde{\nabla} p^{(0)}) + \partial_z v^{(0)} \partial_y(e_3 \cdot \tilde{\nabla} p^{(0)}).$$

Using the fact that $u^{(0)} = -p_y^{(0)}$ and $v^{(0)} = p_x^{(0)}$, the second two terms cancel each other out. The horizontal divergence $(\partial_x, \partial_y, 0)$ of $d_g(\tilde{\nabla} p^{(0)})$ is easy to calculate from the stratification and the divergence free nature of the zero-order flow. We arrive at the equation

$$\left(\partial_t - p_y^{(0)}\partial_x + p_x^{(0)}\partial_y\right)\left(p_{xx}^{(0)} + p_{yy}^{(0)} + (\lambda p_z^{(0)})_z + \beta_0 y\right) = 0$$

after absorbing the β-plane term into the material derivative and defining $\lambda = -\frac{1}{\varrho_z}$. Note that by the assumptions on the density, there exists Λ such that $\frac{1}{\Lambda} \le \lambda \le \Lambda$. We shall use the notation Ψ for the stream function $p^{(0)}$, allowing us to rewrite the system in the familiar form

$$\left(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\right)(\mathcal{L}(\Psi) + \beta_0 y) = 0. \tag{4.5}$$

Consider now the top and bottom $\Omega \times \{0\}$ and $\Omega \times \{h\}$. Let ν denote the unit normal vector on the top and bottom. Considering the equation

$$d_g(-p_z^{(0)}) + w^{(1)}\varrho_z = 0,$$

using that $w^{(1)} \equiv 0$ on the top and bottom, and substituting the notation Ψ for the stream function, we obtain

$$\left(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\right)(\partial_\nu \Psi) = 0. \tag{4.6}$$

4.2.3 The Lateral Boundary

Now consider the sides $\partial\Omega \times [0, h]$ equipped with a horizontal normal vector ν_s. First, the impermeability requires that $\overline{\nabla}^{\perp} p^{(0)} \cdot \nu_s = 0$, implying that $p^{(0)}$ is constant on $\partial\Omega \times \{z\}$. Recalling that the stream function $\Psi = p^{(0)}$, we have that

$$\Psi(t, y, x, z)|_{\{\partial\Omega\times[0,h]\}} = c(t, z) \tag{4.7}$$

for some unknown function $c(t, z)$.

Let us next take the dot product of (4.4) with ν_s. Due to the impermeability of the boundary,

$$(u^{(1)}, v^{(1)}, w^{(1)})^t \cdot \nu_s = 0.$$

In addition,

$$(-p_y^{(1)}, p_x^{(1)}, 0)^t \cdot \nu_s = -(p_x^{(1)}, p_y^{(1)}, 0)^t \cdot \tau$$

where τ is the positively oriented tangent vector perpendicular to ν_s. Then we integrate around the boundary $\partial\Omega \times \{z\} \subset \partial\Omega \times [0, h]$ at a fixed height z. Since $(p_x^{(1)}, p_y^{(1)}, 0)^t$ is a conservative vector field,

$$\int_{\partial\Omega\times\{z\}} (p_x^{(1)}, p_y^{(1)}, 0)^t \cdot \tau \, d\omega = 0.$$

Notice that

$$\beta_0 y(-p_y^{(0)}, p_x^{(0)}, 0)^t - \beta_0(p^{(0)}, 0, 0)^t$$

is also the two-dimensional curl $\overline{\nabla}^{\perp}$ of the scalar field $-\beta_0 y p^{(0)}$. Then we have that

$$\overline{\nabla}^{\perp}(-\beta_0 y p^{(0)}) \cdot \nu_s = \overline{\nabla}(\beta_0 y p^{(0)}) \cdot \tau.$$

As this is also a conservative vector field, the integral of this term around the boundary vanishes as well. Thus we are left with

$$\int_{\partial\Omega\times\{z\}} (d_g \tilde{\nabla} p^{(0)}) \cdot \nu_s \, d\omega = -\int_{\partial\Omega\times\{z\}} (\beta_0 p^{(0)}, 0, 0) \cdot \nu_s \, d\omega. \tag{4.8}$$

Using (4.7) shows that

$$-\int_{\partial\Omega\times\{z\}} (\beta_0 p^{(0)}, 0, 0)\cdot \nu_s \, d\omega$$

is zero. Substituting in the stream function notation and applying the divergence theorem to the nonlinear term on the left hand side of (4.8), we have that

$$\begin{aligned}
\int_{\partial\Omega\times\{z\}} \left(-p_y^{(0)}\partial_x \overline{\nabla} p^{(0)} + p_x^{(0)}\partial_y \overline{\nabla} p^{(0)}\right)\cdot \nu_s \, d\omega &= \int_{\partial\Omega\times\{z\}} \overline{\nabla}\cdot\left(\overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\overline{\nabla}\Psi\right)\cdot\nu_s\, d\omega \\
&= \int_{\Omega\times\{z\}} \overline{\nabla}\overline{\nabla}^{\perp}\Psi : \overline{\nabla}\overline{\nabla}\Psi \, dx\, dy + \int_{\Omega\times\{z\}} \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\overline{\Delta}\Psi\, dx\, dy \\
&= \int_{\Omega\times\{z\}} \overline{\nabla}\cdot\left(\overline{\nabla}^{\perp}\Psi\overline{\Delta}\Psi\right)\, dx\, dy \\
&= \int_{\partial\Omega\times\{z\}} \overline{\Delta}\Psi\left(\overline{\nabla}^{\perp}\Psi\cdot\nu_s\right)\, d\omega \\
&= 0.
\end{aligned}$$

Utilizing once again the notation Ψ for the stream function, (4.8) therefore becomes

$$\frac{\partial}{\partial t}\int_{\partial\Omega\times\{z\}} (\overline{\nabla}\Psi)\cdot\nu_s \, d\omega = 0. \tag{4.9}$$

Collecting (4.5), (4.6), (4.7), and (4.9), we have formally derived the following system:

$$\begin{cases}
\left(\partial_t + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\right)(\mathcal{L}(\Psi)+\beta_0 y) = 0 & \Omega\times[0,h] \\
\left(\partial_t + \overline{\nabla}^{\perp}\Psi\cdot\overline{\nabla}\right)(\partial_\nu \Psi) = 0 & \Omega\times\{0,h\} \\
\frac{\partial}{\partial t}\int_{\partial\Omega\times\{z\}}(\overline{\nabla}\Psi)\cdot\nu_s\, d\omega = 0 & [0,h] \\
\Psi = c(t,z) & \partial\Omega\times[0,h].
\end{cases}$$

4.3 The Elliptic Problem

4.3.1 Building a solution in L^2

In order to show global existence of weak solutions to the time-dependent problem, we first solve the stationary elliptic problem which is transported by the fluid velocity $\overline{\nabla}^{\perp}\Psi$. The elliptic operator is given by $\mathcal{L}$. The boundary conditions for the elliptic problem will be mixed in nature. We first impose a Neumann condition on the top and bottom of $\Omega\times[0,h]$ coming from the transport equation for $\partial_\nu\Psi$. The condition that

$$\Psi(t,x,y,z)|_{\partial\Omega\times[0,h]} = c(t,z)$$

will be structured into the Hilbert space within which we solve the elliptic problem. Finally, the equation

$$\frac{\partial}{\partial t}\int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi\cdot\nu_s\,d\omega = 0$$

means that

$$\int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi(t)\cdot\nu_s\,d\omega = \int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi(0)\cdot\nu_s\,d\omega =: j(z)$$

is determined from the initial data, and thus will be incorporated into the data of the elliptic problem. We now provide a weak formulation of this condition for (QG).

Definition 4.3.1. *Let $T > 0$ be given and $\Psi(t,x,y,z) : [0,T]\times\Omega\times[0,h]$ be such that $\nabla\Psi, \mathcal{L}(\Psi) \in L^\infty([0,T]; L^2(\Omega\times[0,h]))$, and for each time, Ψ has mean value zero. Then we say that Ψ satisfies* (4.2) *weakly if there exists $j_0(z) : [0,h] \to \mathbb{R}$ such that for each compactly supported smooth function $\phi(t,z) : [0,T]\times[0,h] \to \mathbb{R}$,*

$$\int_0^T\int_{\Omega\times[0,h]} \mathcal{L}(\Psi)\phi(t,z) - \Psi\partial_z\left(\lambda\partial_z\phi(t,z)\right)\,dx\,dy\,dz\,dt = \int_0^T\int_0^h \phi(t,z)j_0(z)\,dz\,dt.$$

An integration by parts shows that for smooth functions of time and space, (4.2) is equivalent to Definition 4.3.1. Indeed,

$$\begin{aligned}
&\int_0^T\int_{\Omega\times[0,h]} \mathcal{L}(\Psi)\phi(t,z) - \Psi\partial_z\left(\lambda\partial_z\phi(t,z)\right)\,dx\,dy\,dz\,dt\\
&\qquad = \int_0^T\int_{\Omega\times[0,h]} \mathcal{L}(\Psi)\phi(t,z) - \partial_z\left(\lambda\partial_z\Psi\right)\phi(t,z)\,dx\,dy\,dz\,dt\\
&\qquad = \int_0^T\int_{\Omega\times[0,h]} \left(\partial_{xx}\Psi + \partial_{yy}\Psi\right)\phi(t,z)\,dx\,dy\,dz\,dt\\
&\qquad = \int_0^T\int_0^h\int_{\partial\Omega} \phi(t,z)\overline{\nabla}\Psi\cdot\nu_s\,d\omega\,dz\,dt
\end{aligned}$$

Thus we consider the elliptic problem for the unknown function u with data $f : \Omega\times[0,h]\to\mathbb{R}$, $g : \Omega\times\{0,h\}\to\mathbb{R}$, and $j : [0,h]\to\mathbb{R}$.

$$(E) = \begin{cases}
\mathcal{L}(u) = f & \Omega\times[0,h] \qquad (E1)\\
\partial_\nu u = g & \Omega\times\{0,h\} \qquad (E2)\\
u(x,y,z) = c(z) & \partial\Omega\times[0,h] \qquad (E3)\\
\int_{\partial\Omega\times\{z\}} \overline{\nabla}u\cdot\nu_s = j(z) & [0,h] \qquad (E4).
\end{cases}$$

Let us remark that to formulate (E) variationally, it is not necessary for (f, g, j) to satisfy the compatibility condition Definition 4.1.1. Indeed our construction of approximate solutions will introduce a small error in the condition of Definition 4.1.1 which will vanish in the limit. Thus when we say that u is a solution to (E), we generally mean it in the variational sense of (V) (see (4.12) below). If in addition, (f, g, j) satisifes the compatibility condition so that (V) is equivalent to (E), we shall make note of this. To solve (V) we require a specially constructed Hilbert space.

Definition 4.3.2. *Define H by*

$$H := \{\alpha \in C^\infty\left(\bar{\Omega} \times [0, h]\right) : \quad \int_{\Omega\times[0,h]} \alpha\, dx\, dy\, dz = 0, \quad \alpha|_{\partial\Omega\times[0,h]}(x, y, z) = \alpha(z)\}.$$

Using the notation $\tilde{\nabla} = (\partial_x, \partial_y, \lambda(z)\partial_z)$, equip H with the inner product

$$\langle \alpha, \gamma\rangle_{\mathbb{H}} := \int_{\Omega\times[0,h]} \tilde{\nabla}\alpha \cdot \nabla\gamma\, dx\, dy\, dz.$$

Define the Hilbert space $\mathbb{H}$ as the closure of H under the norm induced by this inner product.

By standard trace inequalities and Poincaré's inequality, we have that for $\gamma \in \mathbb{H}$

$$\|\gamma\|_{H^{\frac{1}{2}}(\partial(\Omega\times[0,h]))} \leq C(\Omega, h)\left(\|\gamma\|_{L^2(\Omega\times[0,h])} + \|\nabla\gamma\|_{L^2(\Omega\times[0,h])}\right) \leq C(\Omega, h, \lambda)\|\gamma\|_{\mathbb{H}} \tag{4.10}$$

We define a bilinear form $B(\alpha, \gamma) : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ and functional $F(\gamma) : \mathbb{H} \to \mathbb{R}$ by

$$B(\alpha, \gamma) = \int_{\Omega\times[0,h]} \tilde{\nabla}\alpha \cdot \nabla\gamma\, dx\, dy\, dz$$

and

$$F(\gamma) = -\int_{\Omega\times[0,h]} f\gamma\, dx\, dy\, dz + \int_{\Omega\times\{0,h\}} \lambda g\gamma\, dx\, dy + \int_0^h j(z)\gamma|_{\partial\Omega\times\{z\}}\, dz.$$

The coercivity and continuity of the bilinear form B is immediate from the assumptions on $\lambda(z)$ and the definition of $\mathbb{H}$. In addition, we have that

$$\begin{aligned}|F(\gamma)| &\leq \|f\|_{L^2(\Omega\times[0,h])}\|\gamma\|_{L^2(\Omega\times[0,h])} + \|\lambda\|_{L^\infty(0,h)}\|g\|_{L^2(\Omega\times\{0,h\})}\|\gamma\|_{L^2(\Omega\times\{0,h\})} \\ &\quad + \|j\|_{\left(H^{\frac{1}{2}}(\partial\Omega\times[0,h])\right)^*}\|\gamma\|_{H^{\frac{1}{2}}(\partial\Omega\times[0,h])} \\ &\leq C(\Omega, h, \lambda)\left(\|f\|_{L^2} + \|g\|_{L^2} + \|j\|_{\left(H^{\frac{1}{2}}\right)^*}\right)\|\gamma\|_{\mathbb{H}} \end{aligned} \tag{4.11}$$

after applying Hölder's inequality and (4.10). Applying the Lax-Milgram theorem, we obtain a unique solution $u \in \mathbb{H}$ to the variational problem

$$B(u, \gamma) = F(\gamma) \qquad \forall \gamma \in \mathbb{H}. \qquad\qquad (V) \tag{4.12}$$

Let us rigorously state the results of the above argument.

Lemma 4.3.1. *For any data $f \in L^2(\Omega \times [0, h])$, $g \in L^2(\Omega \times \{0, h\})$, and $j \in \left(H^{\frac{1}{2}}([0, h])\right)^*$ there exists a unique solution $u \in \mathbb{H}$ to the variational problem (V) with*

$$\|u\|_{\mathbb{H}} \leq C(\Omega, h, \lambda) \left(\|f\|_{L^2} + \|g\|_{L^2} + \|j\|_{\left(H^{\frac{1}{2}}\right)^*} \right).$$

If in addition (f, g, j) verifies the compatibility condition in Definition 4.1.1, then

1. *$(E1)$ is satisfied in the weak sense*
2. *$(E2)$ is satisfied in the weak sense*
3. *$(E3)$ is satisfied pointwise*
4. *$(E4)$ is satisfied weakly. That is, for $\phi \in C^\infty$ depending only on z,*

$$\int_{\Omega \times [0,h]} \mathcal{L}(u)\phi(z) - u\partial_z \left(\lambda \partial_z \phi(z)\right) \, dx \, dy \, dz = \langle j, \phi \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes duality between $\left(H^{\frac{1}{2}}\right)^*$ and $H^{\frac{1}{2}}$.

Proof. The first claim is simply the above construction of u as the solution to the variational problem (V). For (1)-(4), the compatibility condition implies that constant functions γ can be used in the weak formulation, and therefore any C^∞ test function such that $\gamma(x, y, z)|_{\partial\Omega \times [0,h]} = c(z)$ is valid in the weak formulation. Parts (1) and (2) then follow from considering test functions which vanish on the lateral boundary $\partial\Omega \times [0, h]$. Part (3) is a consequence of constructing the solution within $\mathbb{H}$. Finally, (4) follows from noticing that when ϕ depends only on z,

$$B(u, \phi) = \int_{\Omega \times [0,h]} \lambda \partial_z u \partial_z \phi.$$

Rearranging the equality $B(u, \phi) = F(\phi)$ and using (1) finishes the proof. □

4.3.2 Higher Regularity

In order to build weak solutions, the operator which sends a triple (f, g, j) to the solution of the variational problem (V) must map compactly into $\mathbb{H}$. This will be achieved by proving an elliptic regularity theorem which asserts that the solution has strictly more than one derivative in $L^2(\Omega\times[0,h])$. The proof is split up into four preliminary lemmas which correspond to isolating the effects of the compatibility condition, g, f, and j on the regularity of the solution. Specifying a triple of data which does not satisfy Definition 4.1.1 produces a solution by projecting, in an appropriate sense, the data onto the set of compatible data. Analysis of the effect g is direct because solutions to the extension problem on bounded domains Ω can be written down explicitly. Once the Neumann derivative has been removed, we analyze the effects of f and j by reflecting the solution over the boundaries $z = 0, h$ and utilizing the standard difference quotient technique for elliptic regularity. Each step is proved for the special case $\lambda(z) \equiv 1$, i.e. when $\mathcal{L} = \Delta$. The four lemmas are combined in the proof of the following theorem, where we then provide a description of how to adapt the techniques to general smooth λ.

Theorem 4.3.2. *Let $f \in L^2(\Omega \times [0, h])$, $g \in L^2(\Omega \times \{0, h\})$, and $j \in L^2([0, h])$. Let $u \in \mathbb{H}$ be the unique variational solution to (V) guaranteed by Lemma 4.3.1. Then*

$$\|\nabla u\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega, h, \lambda)\left(\|f\|_{L^2(\Omega\times[0,h])} + \|g\|_{L^2(\Omega\times\{0,h\})} + \|j\|_{L^2([0,h])}\right).$$

Before beginning the analysis, we set several notations. Let $\{e_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ be the sequence of eigenfunctions and corresponding eigenvalues for the operator $-\overline{\Delta}$ on Ω with homogenous Dirichlet boundary conditions; that is,

$$\begin{cases} -\overline{\Delta} e_n = \lambda_n e_n & (x, y) \in \Omega \\ e_n = 0 & (x, y) \in \partial\Omega. \end{cases}$$

For $s \geq 0$, define

$$\bar{H}^s(\Omega) = \{g = \sum_n g_n e_n \in L^2(\Omega) : \sum_n \left(\sqrt{\lambda_n}\right)^s g_n e_n \in L^2(\Omega)\}.$$

By duality, we have that

$$\left(\bar{H}^s(\Omega)\right)^* \cong \{\{g_n\}_{n=1}^{\infty} \subset \mathbb{R} : \sum_n \frac{1}{\left(\sqrt{\lambda_n}\right)^{2s}} g_n^2 < \infty\}.$$

Real interpolation of Hilbert spaces H_1, H_2 is defined in the classical way (following the book of Bergh and Lofstrom for example [5]). For non-integer $s \in (-\infty, \infty)$, the Stein-Weiss interpolation theorem (see for example the book of Bergh and Lofstrom [5]) gives that

$$[\bar{H}^{s_1}(\Omega), \bar{H}^{s_2}(\Omega)]_\theta = \bar{H}^s$$

for $s = \theta s_1 + (1-\theta)s_2$ where $s_1, s_2 \in \mathbb{Z}$. When $s = 0$, $\bar{H}^s(\Omega)$ coincides with $L^2(\Omega)$. In general, $\bar{H}^s(\Omega) \subset H^s(\Omega)$ if $H^s(\Omega)$ is defined classically (see for example Constantin and Nguyen [36]).

For $s \in (0,1)$, the fractional Sobolev spaces $H^s(\Omega \times [0,h])$ are defined by

$$H^s(\Omega \times [0,h]) := \left\{ h \in L^2(\Omega \times [0,h]) : \frac{|h(x_1) - h(x_2)|}{|x_1 - x_2|^{\frac{3}{2}+s}} \in L^2\left((\Omega \times [0,h]) \times (\Omega \times [0,h])\right) \right\}.$$

For $s \in \mathbb{N} + (0,1)$, $H^s(\Omega \times [0,h])$ is the subset of $L^2(\Omega \times [0,h])$ for which

$$\frac{|\nabla^{\lfloor s \rfloor}(h(x_1) - h(x_2))|}{|x_1 - x_2|^{\frac{3}{2}+s-\lfloor s \rfloor}} \in L^2\left((\Omega \times [0,h]) \times (\Omega \times [0,h])\right).$$

Classical interpolation results (see for example the work of Triebel [93], [94]) give that

$$[H^{s_1}(\Omega \times [0,h]), H^{s_2}(\Omega \times [0,h])]_\theta = H^s(\Omega \times [0,h])$$

for $s = \theta s_1 + (1-\theta)s_2$.

Lemma 4.3.3 (Effect of the Compatibility Condition). *Let a triple (f,g,j) with $f \in L^2(\Omega \times [0,h])$, $g \in L^2(\Omega \times \{0,h\})$, $j \in L^2(0,h)$ be given. Let u be the solution to the variational problem with data (f,g,j). Then there exists a constant c depending only on*

$$\int_{\Omega \times [0,h]} f, \quad \int_{\Omega \times \{0,h\}} g, \quad \int_0^h j$$

such that $\Delta u = f + c$ and

$$|c| \lesssim \|f\|_{L^2} + \|g\|_{L^2} + \|j\|_{L^2}.$$

Proof. We define an operator $A : L^2(\Omega \times [0,h]) \times L^2(\Omega \times \{0,h\}) \times L^2(0,h) \to \mathbb{R}$ which maps a triple (f,g,j) to a constant $c = A(f,g,j)$. Since $\mathbb{H}$ only contains test functions with mean value zero, given $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega \times [0, h]$, choose a sequence of test functions which is the difference between two sequences of approximate identities centered at $(x_1, y_1, z_1), (x_2, y_2, z_2)$. Using this sequence of test functions in the variational formulation gives that $\Delta u(x_1, y_1, z_1) - \Delta u(x_2, y_2, z_2) = f(x_1, y_1, z_1) - f(x_2, y_2, z_2)$. Therefore, Δu is equal to f up to a constant c, and thus $A(f, g, j) = c$ is well-defined. By the linearity of the variational problem, A is linear. To show that A depends only on the integrals of f, g, and j, let $\bar{f}, \bar{g}, \bar{j}$ be given, each with mean value zero. Then $A(\bar{f}, \bar{g}, \bar{j})$ satisfies the compatibility condition, implying $\Delta \bar{u} = \bar{f}$ in a weak sense, and $A(\bar{f}, \bar{g}, \bar{j}) = 0$. Therefore A depends only on

$$\int_{\Omega \times [0,h]} f, \quad \int_{\Omega \times \{0,h\}} g, \quad \int_0^h j.$$

Now A is a linear map from $\mathbb{R}^3 \to \mathbb{R}$, and is therefore bounded. That is,

$$|A(f, g, j)|^2 \lesssim \Big| \int_{\Omega \times [0,h]} f \Big|^2 + \Big| \int_{\Omega \times \{0,h\}} g \Big|^2 + \Big| \int_0^h j \Big|^2.$$

Applying Hölder's inequality finishes the proof. □

Lemma 4.3.4 (Effect of g). *Consider the equation*

$$\begin{cases} \Delta u = 0 & \Omega \times [0, h] \\ \partial_\nu u = g & \Omega \times \{0, h\} \\ u = 0 & \partial\Omega \times [0, h]. \end{cases}$$

for $g \in \bar{H}^s(\Omega \times \{0, h\})$, $s \geq -\frac{1}{2}$. Then there exists a solution u which satisfies

$$\|\nabla u\|_{H^{s+\frac{1}{2}}(\Omega \times [0,h])} \leq C(\Omega, h) \|g\|_{\bar{H}^s(\Omega \times \{0,h\})}$$

Proof. We begin by assuming that g is smooth so that all calculations with higher derivatives are valid. For arbitrary $g \in \bar{H}^s$, the claim follows from density of smooth functions. By assumption on g, there exist sequences of real numbers $\{t_n\}, \{b_n\}$ such that

$$g(0, x, y) = \sum_n b_n e_n(x, y), \qquad g(h, x, y) = \sum_n t_n e_n(x, y).$$

Define

$$u = \sum_n \left\{ \frac{t_n}{\sqrt{\lambda_n}} \frac{\cosh(z\sqrt{\lambda_n})}{\sinh(h\sqrt{\lambda_n})} + \frac{b_n}{\sqrt{\lambda_n}} \frac{\cosh((z - h)\sqrt{\lambda_n})}{\sinh(h\sqrt{\lambda_n})} \right\} e_n(x, y).$$

Using that $\sinh(0) = 0$ and $(\sinh)'' = (\cosh)' = \sinh$, we have that

$$-\frac{\partial}{\partial z} u|_{z=0} = g|_{z=0}$$

and

$$\frac{\partial}{\partial z} u|_{z=h} = g|_{z=h}.$$

In addition, it is immediate that

$$\partial_{zz} u = -\overline{\Delta} u,$$

and therefore $\Delta u \equiv 0$. Since

$$\cosh(z\sqrt{\lambda_n}) \approx \sinh(h\sqrt{\lambda_n}) \approx e^{z\sqrt{\lambda_n}}$$

as $n \to \infty$, we have that

$$(-\overline{\Delta})^{\frac{1}{2}} u \approx \partial_z u \in L^\infty([0,h]; L^2(\Omega)) \subset L^2(\Omega \times [0,h]).$$

Using the well-known fact that $\bar{H}^1(\Omega) = H_0^1(\Omega)$, we have that

$$\|\nabla u\|_{L^2(\Omega\times[0,h])} \leq C(\Omega, h)\|g\|_{L^2(\Omega\times\{0,h\})},$$

and thus u is a well-defined function in $\Omega \times [0,h]$ which solves the desired equation.

To sharpen this bound and obtain higher regularity estimates, we split the sum into four pieces corresponding to the four pieces of

$$\cosh(z\sqrt{\lambda_n}) = \frac{e^{z\sqrt{\lambda_n}} + e^{-z\sqrt{\lambda_n}}}{2}, \qquad \cosh((z-h)\sqrt{\lambda_n}) = \frac{e^{(z-h)\sqrt{\lambda_n}} + e^{-(z-h)\sqrt{\lambda_n}}}{2}.$$

Define

$$\tilde{t}_n := \frac{t_n}{\sinh(h\sqrt{\lambda_n})} e^{h\sqrt{\lambda_n}}, \qquad \tilde{g} := \sum_n \tilde{t}_n e_n$$

so that

$$\|\tilde{g}\|_{\bar{H}^s(\Omega)} \leq C(\Omega)\|g\|_{\bar{H}^s(\Omega\times\{0,h\})}.$$

Then

$$\tilde{u} := \sum_n e^{(z-h)\sqrt{\lambda_n}} \frac{\tilde{t}_n}{\sqrt{\lambda_n}}$$

is the solution to

$$\begin{cases} \Delta \tilde{u} = 0 & \Omega \times (-\infty, h] \\ \partial_\nu \tilde{u} = \tilde{g} & \Omega \times \{h\} \\ \tilde{u}(x,y,z) = 0 & \partial\Omega \times (-\infty, h]. \end{cases}$$

Now we can write that

$$\begin{aligned} \|\nabla \tilde{u}\|_{\bar{H}^{s+\frac{1}{2}}(\Omega\times(-\infty,h))} &= \int_{\Omega\times[0,h]} \nabla\left((-\overline{\Delta})^{\frac{1}{2}(s+\frac{1}{2})}\tilde{u}\right)\cdot\nabla\left((-\overline{\Delta})^{\frac{1}{2}(s+\frac{1}{2})}\tilde{u}\right) \\ &= \int_{\Omega\times\{h\}} \partial_\nu(-\overline{\Delta})^{\frac{1}{2}(s+\frac{1}{2})}\tilde{u}(-\overline{\Delta})^{\frac{1}{2}(s+\frac{1}{2})}\tilde{u} \\ &= \|\partial_\nu \tilde{u}\|_{\bar{H}^s(\Omega)} \\ &\leq C(\Omega,h)\|g\|_{\bar{H}^s(\Omega\times\{0,h\})} \end{aligned} \tag{4.13}$$

Arguing in a similar fashion for the other parts of the infinite sum, we conclude that

$$\|\nabla u\|_{\bar{H}^{s+\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega,h)\|g\|_{\bar{H}^s(\Omega\times\{0,h\})}.$$

If $s+\frac{1}{2}\in\mathbb{N}$, noticing that $(\partial_z)^{s+\frac{1}{2}}u \approx (-\overline{\Delta})^{\frac{s+\frac{1}{2}}{2}}u$, we have that

$$\|\nabla u\|_{H^{s+\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega,h)\|g\|_{\bar{H}^s(\Omega\times\{0,h\})}. \tag{4.14}$$

As noted above, for non-integer $s\in(-\frac{1}{2},\infty)$, the Stein-Weiss interpolation theorem gives that

$$[\bar{H}^{s_1}(\Omega), \bar{H}^{s_2}(\Omega)]_\theta = \bar{H}^s(\Omega)$$

for $s = \theta s_1 + (1-\theta)s_2$, and interpolation of Hilbert-Sobolev spaces on Lipschitz domains gives that

$$[H^{s_1+\frac{1}{2}}(\Omega\times[0,h]), H^{s_2+\frac{1}{2}}(\Omega\times[0,h])]_\theta = H^{s+\frac{1}{2}}(\Omega\times[0,h]).$$

Interpolation of (4.14) then concludes the proof of the lemma. □

In the following two lemmas, we address the effects of f and j. While the solutions we consider are only variational a priori, for the sake of clarity we write each PDE using classical notation rather than the variational form.

Lemma 4.3.5 (Effect of f)**.** *Let $u \in \mathbb{H}$ be a variational solution to*

$$\begin{cases} \Delta u = f & \Omega \times [0,h] \\ \partial_\nu u = 0 & \Omega \times \{0,h\} \\ u(x,y,z) = c(z) & \partial\Omega \times [0,h] \\ \int_{\partial\Omega\times\{z\}} \overline{\nabla} u \cdot \nu_s = 0 & [0,h]. \end{cases}$$

for data $f \in L^2(\Omega \times [0,h])$. Then

$$\|u\|_{H^2(\Omega\times[0,h])} \leq C(\Omega,h)\|f\|_{L^2(\Omega\times[0,h])}.$$

Proof. Formally, the assumptions that

$$\partial_\nu u \equiv 0, \qquad \int_{\partial\Omega\times\{z\}} \overline{\nabla} u \cdot \nu_s \equiv 0$$

give

$$\int_{\Omega\times[0,h]} \nabla(\partial_z u) \cdot \nabla(\partial_z u) = -\int_{\Omega\times[0,h]} \partial_z u \Delta(\partial_z u) = \int_{\Omega\times[0,h]} \partial_{zz} u f,$$

implying that

$$\|\partial_{zz} u\|_{L^2(\Omega\times[0,h])} \leq C\|f\|_{L^2(\Omega\times[0,h])}. \tag{4.15}$$

Regularity of $\overline{\Delta} u$ would then follow from the equality

$$\overline{\Delta} u = f + A(f,0,0) - \partial_{zz} u.$$

Then we can write that for fixed z,

$$\begin{cases} \overline{\Delta} u = f + A(f,0,0) - \partial_{zz} u & \Omega \times \{z\} \\ u = c(z) & \partial\Omega \times \{z\}. \end{cases}$$

Applying classical elliptic regularity theory z by z shows then that $\partial_{xy} u, \partial_{xx} u, \partial_{yy} u \in L^2(\Omega \times [0,h])$. Thus it remains to rigorously show (4.15).

Define

$$u_E(x,y,z) = \begin{cases} u(x,y,z) & z \in [0,h] \\ u(x,y,-z) & z \in [-h,0] \end{cases}$$

and define f_E similarly. Let $\eta(z)$ be a smooth cutoff function depending only on z such that $\eta \equiv 1$ for all $z \in [-\frac{h}{2}, \frac{h}{2}]$ and η is compactly supported in $[-\frac{3h}{4}, \frac{3h}{4}]$. Define the difference quotient operator

$$T_\epsilon \phi = \frac{\phi(x,y,z+\epsilon) - \phi(x,y,z)}{\epsilon}.$$

Then we can write

$$\begin{aligned}
-\int_{\Omega\times[-h,h]} \nabla u_E \cdot \nabla \left(T_{-\epsilon}(\eta^2 T_\epsilon u_E)\right) &= \int_{\Omega\times[-h,h]} \nabla(T_\epsilon u_E)\cdot\nabla(\eta^2 T_\epsilon u_E) \\
&= \int_{\Omega\times[-h,h]} \nabla(T_\epsilon u_E)\cdot\nabla(T_\epsilon u_E)\eta^2 \\
&\quad + \int_{\Omega\times[-h,h]} \nabla(T_\epsilon u_E)\cdot\nabla\eta(2\eta T_\epsilon u_E) \\
&\geq \int_{\Omega\times[-\frac{h}{2},\frac{h}{2}]} |\nabla(T_\epsilon u_E)|^2 + \int_{\Omega\times[-h,h]} \nabla(T_\epsilon u_E)\cdot\nabla\eta(2\eta T_\epsilon u_E).
\end{aligned}$$

Rearranging, we have that

$$\begin{aligned}
\int_{\Omega\times[-\frac{h}{2},\frac{h}{2}]} |\nabla(T_\epsilon u_E)|^2 &\leq -\int_{\Omega\times[-h,h]} \nabla(T_\epsilon u_E)\cdot\nabla\eta(2\eta T_\epsilon u_E) \\
&\quad - \int_{\Omega\times[-h,h]} \nabla u_E \cdot \nabla\left(T_{-\epsilon}(\eta^2 T_\epsilon u_E)\right) \\
&:= I + II.
\end{aligned}$$

Examining I, we have that

$$\begin{aligned}
I &\leq C(\eta)\|\nabla(T_\epsilon u_E)\|_{L^2(\Omega\times[-h,h])}\|\nabla u_E\|_{L^2(\Omega\times[-h,h])} \\
&\leq C(\eta)\left(\frac{1}{8}\|\nabla(T_\epsilon u_E)\|^2_{L^2(\Omega\times[-h,h])} + 4\|f\|^2_{L^2(\Omega\times[0,h])}\right) \qquad (4.16)
\end{aligned}$$

Moving to II, we have that

$$\begin{aligned}
II &= \int_{\Omega\times[-h,h]} T_{-\epsilon}(\eta^2(T_\epsilon u_E))f_E \\
&\leq C(\eta)\left(\frac{1}{8}\|T_{-\epsilon}T_\epsilon u_E\|^2_{L^2(\Omega\times[-h,h])} + 4\|f\|^2_{L^2(\Omega\times[0,h])}\right) \qquad (4.17)
\end{aligned}$$

Combining (4.16) and (4.17) and repeating the argument but this time with a reflection over $z = h$, it follows that

$$\int_{\Omega\times[0,h]} |\nabla(T_\epsilon u)|^2 + \int_{\Omega\times[0,h]} |T_{-\epsilon}(T_\epsilon u)|^2 \leq 100C(\eta)\|f\|_{L^2(\Omega\times[0,h])}.$$

The uniformity of this inequality in ϵ allows us to pass to a weak limit as $\epsilon \to 0$ to conclude that

$$\|\nabla(\partial_z u)\|_{L^2(\Omega\times[0,h])} \leq C(\Omega,\eta)\|f\|^2_{L^2(\Omega\times[0,h])}.$$

Regularity of $\partial_{xx}u$, $\partial_{xy}u$, and $\partial_{yy}u$ follows as described before, finishing the proof of the lemma. $\square$

Lemma 4.3.6 (Effect of j)**.** *Let $u \in \mathbb{H}$ be a variational solution to*

$$\begin{cases} \Delta u = 0 & \Omega\times[0,h] \\ \partial_\nu u = 0 & \Omega\times\{0,h\} \\ u(x,y,z) = c(z) & \partial\Omega\times\{z>0\} \\ \int_{\partial\Omega\times\{z\}} \overline{\nabla} u\cdot\nu_s = j(z) & [0,h]. \end{cases}$$

for data $j \in H^s([0,h])$, $s \in [-\frac{1}{2},\frac{1}{2}]$. Then

$$\|\nabla u\|_{H^{(s+\frac{1}{2})}(\Omega\times[0,h])} \leq C(\Omega,h)\|j\|_{H^s([0,h])}.$$

Proof. The case $s = -\frac{1}{2}$ is the content of Lemma 4.3.1. We shall prove the case $s = \frac{1}{2}$ by hand and deduce the intermediate cases by interpolation.

The proof for $s = \frac{1}{2}$ follows closely that of Lemma 4.3.5. Define u_E and j_E on $[-h,h]$ by reflection as before, and define T_ϵ and η similarly as well. In addition, let $\phi_\epsilon(z)$ be a one dimensional, smooth, even mollifier supported on a ball of radius ϵ around 0. Note $c'(z)$ is yet not well defined as ∇u only belongs to $L^2([0,h])$ for now. However, c should satisfy $c'(0) = c'(h) = 0$, and we shall mollify our test function in z to take advantage of this. Thus we choose our test function to be

$$\phi_\epsilon * T_{-\epsilon}\left(\eta^2 T_\epsilon(u_E * \phi_\epsilon)\right)$$

Then we can write

$$
\begin{aligned}
-\int_{\Omega\times[-h,h]} \nabla u_E \cdot \nabla \left(\phi_\epsilon * T_{-\epsilon}(\eta^2 T_\epsilon(u_E*\phi_\epsilon))\right) &= \int_{\Omega\times[-h,h]} \nabla(T_\epsilon(u_E*\phi_\epsilon))\cdot\nabla(\eta^2 T_\epsilon(u_E*\phi_\epsilon)) \\
&= \int_{\Omega\times[-h,h]} \nabla(T_\epsilon(u_E*\phi_\epsilon))\cdot\nabla(T_\epsilon(u_E*\phi_\epsilon))\eta^2 \\
&\quad + \int_{\Omega\times[-h,h]} \nabla(T_\epsilon(u_E*\phi_\epsilon))\cdot\nabla\eta(2\eta T_\epsilon(u_E*\phi_\epsilon)) \\
&\geq \int_{\Omega\times[-\frac{h}{2},\frac{h}{2}]} |\nabla(T_\epsilon(u_E*\phi_\epsilon))|^2 \\
&\quad + \int_{\Omega\times[-h,h]} \nabla(T_\epsilon(u_E*\phi_\epsilon))\cdot\nabla\eta(2\eta T_\epsilon(u_E*\phi_\epsilon)).
\end{aligned}
$$

Rearranging, we have that

$$
\begin{aligned}
\int_{\Omega\times[-\frac{h}{2},\frac{h}{2}]} |\nabla(T_\epsilon(u_E*\phi_\epsilon))|^2 &\leq -\int_{\Omega\times[-h,h]} \nabla(T_\epsilon(u_E*\phi_\epsilon))\cdot\nabla\eta(2\eta T_\epsilon(u_E*\phi_\epsilon)) \\
&\quad - \int_{\Omega\times[-h,h]} \nabla(u_E*\phi_\epsilon)\cdot\nabla\left(T_{-\epsilon}(\eta^2 T_\epsilon(u_E*\phi_\epsilon))\right) \\
&:= I + II.
\end{aligned}
$$

Examining I, we have that

$$
\begin{aligned}
I &\leq C(\eta)\|\nabla(T_\epsilon(u_E*\phi_\epsilon))\|_{L^2(\Omega\times[-h,h])}\|\nabla(u_E*\phi_\epsilon)\|_{L^2(\Omega\times[-h,h])} \\
&\leq C(\eta)\left(\frac{1}{8}\|\nabla(T_\epsilon(u_E*\phi_\epsilon))\|^2_{L^2(\Omega\times[-h,h])} + 4\|j\|^2_{L^2([0,h])}\right)
\end{aligned}
\tag{4.18}
$$

Before examining II, notice that due to the compact support of η in $[-h,h]$, we can assume without loss of generality that $u_E*\phi_\epsilon|_{\partial\Omega\times[0,h]}$ and $j_E*\phi_\epsilon$ are smooth, compactly supported functions on $[-h,h]$ and therefore can be expanded in Fourier series with coefficients $\hat{u}(k)\hat{\phi}_\epsilon(k)$ and $\hat{j}(k)\hat{\phi}_\epsilon(k)$, respectively. Note also that since T_ϵ ignores constants, we can assume without loss of generality that $\hat{u}(0)=\hat{j}(0)=0$, ensuring that fractional laplacians (as Fourier multipliers) of u_E and j_E are well-defined on $[0,h]$. Furthermore, since $(c_E*\phi)'(z)$ vanishes at 0, the reflected function $u_E*\phi(z)|_{\partial\Omega\times[0,h]}$ belongs to $H^2([-h,h])$. In addition,

$|\hat{\phi}_\epsilon(k)| \leq 1$ for all k and converges to 1 as $\epsilon \to 0$. Then we can write

$$\begin{aligned}
II &= \int_{-h}^{h} (\eta^2 T_\epsilon(u_E * \phi_\epsilon)) T_\epsilon(j_E * \phi_\epsilon) \\
&= \sum_{k=-\infty}^{\infty} \left(\widehat{(T_\epsilon j_E)}(k) \frac{1}{k^{\frac{1}{2}}} \right) \left(\widehat{(\eta^2 T_\epsilon u_E)}(k) k^{\frac{1}{2}} \right) \hat{\phi}_\epsilon(k)^2 \\
&\leq C(\eta) \left(\frac{1}{8} \|T_\epsilon u_E\|^2_{H^{\frac{1}{2}}(\partial\Omega \times [-h,h])} + 4 \|T_\epsilon (-\overline{\Delta})^{-\frac{1}{4}} j_E\|^2_{L^2([-h,h])} \right) \\
&\leq C(\eta, \Omega) \left(\frac{1}{8} \|\nabla (T_\epsilon u_E)\|^2_{L^2(\Omega \times [-h,h])} + 4 \|j_E\|^2_{H^{\frac{1}{2}}([-h,h])} \right).
\end{aligned} \tag{4.19}$$

The last line follows from applying (4.10) to $T_\epsilon u_E$ and noticing that

$$\|T_\epsilon (-\overline{\Delta})^{-\frac{1}{4}} j_E\|_{L^2} \approx \|(-\overline{\Delta})^{\frac{1}{2}} (-\overline{\Delta})^{-\frac{1}{4}} j_E\|_{L^2} \approx \|j_E\|_{H^{\frac{1}{2}}}.$$

Combining (4.18) and (4.19) and repeating the argument but this time with a reflection over $z = h$, it follows that

$$\int_{\Omega \times [0,h]} |\nabla (T_\epsilon u * \phi_\epsilon)|^2 \leq 100 C(\eta, \Omega) \|j\|^2_{H^{\frac{1}{2}}([0,h])}.$$

The uniformity of this inequality in ϵ allows us to pass to a weak limit as $\epsilon \to 0$ to conclude that

$$\|\nabla (\partial_z u)\|_{L^2(\Omega \times [0,h])} \leq C(\Omega, \eta) \|j\|^2_{H^{\frac{1}{2}}([0,h])}.$$

Regularity of $\partial_{xx} u$, $\partial_{xy} u$, and $\partial_{yy} u$ follows as for Lemma 4.3.5, finishing the case $s = \frac{1}{2}$. The intermediate cases follow again from interpolation. $\square$

We can now prove Theorem 4.3.2.

Proof of Theorem 4.3.2. We begin with $\lambda \equiv 1$, in which case (V) is given by

$$\begin{cases} \Delta u = f & \Omega \times [0,h] \\ \partial_\nu u = g & \Omega \times \{0,h\} \\ u(x,y,z) = c(z) & \partial\Omega \times [0,h] \\ \int_{\partial\Omega \times \{z\}} \overline{\nabla} u \cdot \nu_s = j(z) & [0,h]. \end{cases} \tag{V}$$

First, apply Lemma 4.3.4 to build a solution u_1 to

$$\begin{cases} \Delta u_1 = 0 & \Omega \times [0,h] \\ \partial_\nu u_1 = g & \Omega \times \{0,h\} \\ u_1(x,y,z) = 0 & \partial\Omega \times [0,h] \end{cases}$$

which satisfies

$$\|\nabla u_1\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega,h)\|g\|_{L^2(\Omega\times\{0,h\})}.$$

Now choose c_1 such that $\tilde{u}_1 = u_1 + c_1$ has mean value zero on $\Omega\times[0,h]$; then

$$\|\nabla \tilde{u}_1\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} = \|\nabla u_1\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega,h)\|g\|_{L^2(\Omega\times\{0,h\})}. \tag{4.20}$$

By the trace estimate (4.10),

$$j_1(z) := \int_{\partial\Omega\times\{z\}} \overline{\nabla}\tilde{u}_1\cdot\nu_s$$

is well-defined in $L^2([0,h])$ and satisfies

$$\|j_1\|_{L^2([0,h])} \leq C(\Omega,h)\|\nabla\tilde{u}_1\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega,h)\|g\|_{L^2(\Omega\times\{0,h\})}.$$

Therefore, $\tilde{u}_1$ is the unique variational solution to

$$\begin{cases} \Delta\tilde{u}_1 = 0 & \Omega\times[0,h] \\ \partial_\nu\tilde{u}_1 = g & \Omega\times\{0,h\} \\ \tilde{u}_1(x,y,z) = c_1 & \partial\Omega\times[0,h] \\ \int_{\partial\Omega\times\{z\}}\overline{\nabla}\tilde{u}_1\cdot\nu_s = j_1(z) & [0,h]. \end{cases}$$

Now define $u_2 := u - \tilde{u}_1$; u_2 is then the unique variational solution to

$$\begin{cases} \Delta u_2 = f & \Omega\times[0,h] \\ \partial_\nu u_2 = 0 & \Omega\times\{0,h\} \\ u_2(x,y,z) = c_2(z) & \partial\Omega\times[0,h] \\ \int_{\partial\Omega\times\{z\}}\overline{\nabla}u_2\cdot\nu_s = j(z) - j_1(z) & [0,h]. \end{cases}$$

Define u_3 to as the unique variational solution to

$$\begin{cases} \Delta u_3 = f & \Omega\times[0,h] \\ \partial_\nu u_3 = 0 & \Omega\times\{0,h\} \\ u_3(x,y,z) = c_3(z) & \partial\Omega\times[0,h] \\ \int_{\partial\Omega\times\{z\}}\overline{\nabla}u_3\cdot\nu_s = 0 & [0,h] \end{cases}$$

and u_4 as the unique variational solution to

$$\begin{cases} \Delta u_4 = 0 & \Omega\times[0,h] \\ \partial_\nu u_4 = 0 & \Omega\times\{0,h\} \\ u_4(x,y,z) = c_4(z) & \partial\Omega\times[0,h] \\ \int_{\partial\Omega\times\{z\}}\overline{\nabla}u_4\cdot\nu_s = j(z) - j_1(z) & [0,h] \end{cases}$$

so that $u_2 = u_3 + u_4$. Applying Lemma 4.3.5 to u_3 and Lemma 4.3.6 to u_4, we conclude that

$$\|\nabla u_2\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega, h) \left(\|f\|_{L^2(\Omega\times[0,h])} + \|j - j_1\|_{L^2([0,h])}\right). \tag{4.21}$$

Combining (4.20) and (4.21), we conclude that

$$\|\nabla u\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} = \|\nabla(\tilde{u}_1 + u_2)\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega, h) \left(\|f\|_{L^2} + \|g\|_{L^2} + \|j\|_{L^2}\right).$$

We now sketch a proof of how to adapt the argument for arbitrary smooth λ satisfying $\frac{1}{\Lambda} < \lambda < \Lambda$. Let ϕ_1, ϕ_2, ϕ_3 be smooth functions of z such that

$$\phi_1 + \phi_2 + \phi_3 \equiv 1 \qquad \forall z \in [0, h]$$

and

$$\phi_1 \in C_c^\infty(-\delta, 2\delta), \qquad \phi_2 \in C_c^\infty(\delta, h - \delta), \qquad \phi_3 \in C_c^\infty(h - 2\delta, h + \delta)$$

for δ to be chosen later. Because the proofs of Lemma 4.3.5 and Lemma 4.3.6 rely only on the variational structure, the difference quotient technique applies as well to general elliptic operators in divergence form (see for example sections 6.3 or 8.3 of Evans [48]). Since $\partial_\nu(\phi_2 u) \equiv 0$, it follows that $\phi_2 u \in H^{\frac{3}{2}}(\Omega \times [\delta, h - \delta])$.

We focus now on $\phi_1 u$; the argument for $\phi_3 u$ is similar. The goal is to perform a change of variables in z such that the elliptic operator after changing variables is given by the standard Laplacian plus lower order terms depending on the change of variables. By writing

$$\partial_z(\lambda \partial_z u) = \lambda \partial_{zz} u + \partial_z \lambda \partial_z u,$$

notice that we can absorb the first order term $\partial_z \lambda \partial_z u$ into the right hand side, which we rename $\tilde{f}$. Then consider the ordinary differential equation

$$\begin{cases} \theta'(z') = \sqrt{\lambda(\theta(z'))} & z \in [0, \delta'] \\ \theta(0) = 0. \end{cases}$$

By the Cauchy-Lipschitz theorem, for δ' small enough there exists a unique smooth solution θ which, by the positivity of λ, is a bijection between $[0, \delta']$ and $[0, \theta(\delta')]$. Choose $\delta < \frac{\theta(\delta')}{2}$. Then

$$\begin{aligned} \partial_{z'z'}(u(x, y, \theta(z'))) &= u_{33}(x, y, \theta(z'))(\theta'(z))^2 + u_3(x, y, \theta(z'))\theta''(z') \\ &= u_{33}(x, y, \theta(z'))\lambda(\theta(z')) + u_3(x, y, \theta(z'))\theta''(z'). \end{aligned}$$

Absorbing the second term $u_3(x, y, \theta(z'))\theta''(z')$ into the right hand side, (up to the effect of the localization ϕ_1) the elliptic equation becomes

$$\overline{\Delta}(u \circ \theta) + \partial_{z'z'}(u \circ \theta) = \tilde{f} \circ \theta - (u_3 \circ \theta)\theta'',$$

and we can repeat the original argument to show that $\phi_1 u \in H^{\frac{3}{2}}(\Omega \times [0, 2\delta])$. Repeating the argument for $\phi_3 u$ and summing finishes the proof. $\square$

4.4 Proof of Theorem 4.1

4.4.1 Approximate solutions

First, we adjust the initial data and forcing terms. Let η_ϵ be a standard $\mathbb{R}^3$ mollifier supported in a ball of radius ϵ. Define the extension of f to $\mathbb{R}^3$ by

$$f_E(x, y, z) = \begin{cases} f_0(x, y, z) & (x, y, z) \in \Omega \times [0, h] \\ 0 & \text{otherwise,} \end{cases}$$

and mollify by setting $f_\epsilon := f_E * \eta_\epsilon$. After similarly extending $a_L(t)$ to $\mathbb{R}^3$ and $g, a_\nu(t)$ to $\mathbb{R}^2 \times \{0, h\}$ by zero and mollifying (time by time for the forcing terms), we obtain spatially smooth (for example $a_{L,\epsilon} \in L^1([0, T]; C^k(\mathbb{R}^3))$ for any k) sequences of functions such that the following convergences hold:

$$f_\epsilon \to f_0 \quad \text{in} \quad L^2(\Omega \times [0, h])$$

$$g_\epsilon \to g_0 \quad \text{in} \quad L^2(\Omega \times \{0, h\})$$

$$a_{L,\epsilon} \to a_L \quad \text{in} \quad L^1\left([0, T]; L^2(\Omega \times [0, h])\right)$$

$$a_{\nu,\epsilon} \to a_\nu \quad \text{in} \quad L^1\left([0, T]; L^2(\Omega \times \{0, h\})\right).$$

We define the approximate (QG) solution operators $S_\epsilon : C([0, T]; \mathbb{H}) \to C([0, T]; \mathbb{H})$ for $\epsilon > 0$ in several steps. These operators shall provide solutions to linear transport equations with mollified velocity fields.

Step 1: Let $P \in C([0, T]; \mathbb{H})$, and let $c(z)$ be the lateral boundary values of P as usual. We extend $P(t)$ to $\mathbb{R}^3$ for each time in a way which allows for a simple construction of a smooth, stratified velocity field from $\overline{\nabla}^{\perp} P$ which is supported in a small neighborhood of $\Omega \times [0,h]$.

$$P_e(x,y,z) = \begin{cases} P(x,y,z) & (x,y,z) \in \Omega \times [0,h] \\ c(z) & (x,y,z) \in \Omega^{\mathsf{C}} \times [0,h] \end{cases}$$

and

$$P_E(x,y,z) = \begin{cases} P_e(x,y,z) & (x,y,z) \in \mathbb{R}^2 \times [0,h] \\ P_e(x,y,0) & (x,y,z) \in \mathbb{R}^2 \times [-\infty,0] \\ P_e(x,y,h) & (x,y,z) \in \mathbb{R}^2 \times [h,\infty]. \end{cases}$$

Mollify P_E by setting $P_\epsilon := P_E * \eta_\epsilon$.

Step 2: Consider the transport equations for F_ϵ and G_ϵ given by

$$\begin{cases} \left(\partial_t + \overline{\nabla}^{\perp} P_\epsilon \cdot \overline{\nabla}\right)(F_\epsilon + \beta_0 y) = a_{L,\epsilon} & \mathbb{R}^2 \times [0,h] \times [0,\infty) \\ \left(\partial_t + \overline{\nabla}^{\perp} P_\epsilon \cdot \overline{\nabla}\right) G_\epsilon = a_{\nu,\epsilon} & \mathbb{R}^2 \times \{0,h\} \times [0,\infty) \\ F_\epsilon = f_\epsilon & t = 0 \\ G_\epsilon = g_\epsilon & t = 0. \end{cases}$$

Since the initial data, forcing terms, and velocity fields are all smooth, we can produce global in time solutions F_ϵ and G_ϵ by the method of characteristics. Notice that F_ϵ and G_ϵ are defined for $(x,y) \in \mathbb{R}^2$ but supported in a neighborhood of order ϵ around Ω.

Step 3: At each time $t \geq 0$, apply Lemma 4.3.1 to define $Q_\epsilon(t)$ as the solution to

$$\begin{aligned} B(Q_\epsilon(t), v) &:= \int_{\Omega \times [0,h]} \tilde{\nabla} Q_\epsilon(t) \cdot \nabla v \\ &= -\int_{\Omega \times [0,h]} -F_\epsilon(t) v + \int_{\Omega \times \{0,h\}} \lambda G_\epsilon(t) v + \int_0^h v|_{\partial\Omega \times [0,h]} j_0. \\ &=: F(v) \end{aligned}$$

Define $S_\epsilon(P) := Q_\epsilon$. We remark that because F_ϵ and G_ϵ are defined as solutions to transport equations for $(x,y) \in \mathbb{R}^2$ rather than Ω, the compatibility condition is lost. However, Lemma 4.3.1 still produces a solution to the abstract variational problem, and we will recover the compatibility condition in the limit.

In search of fixed points, we will show that the operators $\{S_\epsilon\}_{\epsilon>0}$ are compact, continuous operators from $C([0,T];\mathbb{H})$ to itself with bounded range. Continuity of the operators results from examining the characteristics of the mollified transport equations, while the proof of compactness will require Theorem 4.3.2 and the Aubin-Lions lemma. We split the argument into three lemmas.

Lemma 4.4.1 (Continuity)**.** *The operator S_ϵ is continuous from $C([0,T];\mathbb{H})$ to itself, with modulus of continuity dependent on ϵ.*

Proof. Let

$$P_n \to P \quad \text{in} \quad C([0,T];\mathbb{H}).$$

Define $S_\epsilon(P_n) := Q_{n,\epsilon}$. Using the notation from the construction of the operators S_ϵ, let $F_{n,\epsilon}$ and $G_{n,\epsilon}$ be the solutions to the transport equations with mollified velocity fields $\overline{\nabla}^\perp P_{n,\epsilon}$. Applying Lemma 4.3.1, for fixed $t \in [0,T]$,

$$\|(Q_{n_1,\epsilon} - Q_{n_2,\epsilon})(t)\|_{\mathbb{H}} \lesssim \Big(\|(F_{n_1,\epsilon} - F_{n_2,\epsilon})(t)\|_{L^2(\Omega\times[0,h])} + \|(G_{n_1,\epsilon} - G_{n_2,\epsilon})(t)\|_{L^2(\Omega\times\{0,h\})} \Big).$$

Therefore, it suffices to show that

$$\sup_{t\in[0,T]} \left\{ \|(F_{n_1,\epsilon} - F_{n_2,\epsilon})(t)\|_{L^2(\Omega\times[0,h])} + \|(G_{n_1,\epsilon} - G_{n_2,\epsilon})(t)\|_{L^2(\Omega\times\{0,h\})} \right\} \to 0 \tag{4.22}$$

as $n_1, n_2 \to \infty$.

First, notice that due to the mollification, given $k \in \mathbb{N}$, there exist constants $C(\epsilon,k)$ depending on ϵ, k such that

$$\|\overline{\nabla}^\perp (P_{n_1,\epsilon} - P_{n_2,\epsilon})\|_{L^\infty\left([0,T];C^k(\Omega\times[0,h])\right)} \leq C(\epsilon,k)\|P_{n_1} - P_{n_2}\|_{C([0,T];\mathbb{H})}. \tag{4.23}$$

Fix $(t,x,y,z) \in [0,T]\times\mathbb{R}^2\times[0,h]$, and let Γ_{n_i} for $i = 1,2$ solve

$$\begin{cases} \dot{\Gamma}_{n_i}(s) = \overline{\nabla}^\perp P_{n_i}(s, \Gamma_{n_i}(s)) & s \in [0,t] \\ \Gamma_{n_i}(t) = (x,y,z) \end{cases}$$

Then

$$F_{n_i}(t,x,y,z) = f_\epsilon(\Gamma_{n_i}(t)) + \int_0^t a_{L,\epsilon}(\Gamma_{n_i}(s))\,ds,$$

and

$$F_{n_1}(t,x,y,z) - F_{n_2}(t,x,y,z) = f_\epsilon(\Gamma_{n_1}(t)) - f_\epsilon(\Gamma_{n_2}(t)) + \int_0^t a_{L,\epsilon}(\Gamma_{n_1}(s)) - a_{L,\epsilon}(\Gamma_{n_2}(s))\,ds.$$

Applying (4.23) and using the smoothness of f_ϵ and $a_{L,\epsilon}$ shows that as $n_1, n_2 \to \infty$, $F_{n_1}(t,x,y,z) - F_{n_2}(t,x,y,z)$ converges to 0 uniformly for $(t,x,y,z) \in [0,T] \times \mathbb{R}^2 \times [0,h]$. Arguing similarly for $G_{n_1,\epsilon}, G_{n_2,\epsilon}$ then shows (4.22). □

Lemma 4.4.2 (Time Derivative Bounds)**.** *Let $P \in C([0,T];\mathbb{H})$ with mollified velocity field P_ϵ, and put $S_\epsilon(P) := Q_\epsilon$. Consider $\tilde{\nabla}Q_\epsilon(t)$ as an element of $\mathbb{H}^*$ acting by the rule*

$$v \to \langle Q(t), v\rangle_{\mathbb{H}} \qquad \forall v \in \mathbb{H}.$$

Then $\partial_t\tilde{\nabla}Q_\epsilon$ is a bounded linear functional in $L^\infty([0,T];(\mathbb{H}\cap H^3(\Omega\times[0,h]))^*)$ and

$$\|\partial_t\tilde{\nabla}Q_\epsilon\|_{L^\infty([0,T];(\mathbb{H}\cap H^3(\Omega\times[0,h]))^*)} \leq C\left(f_0, g_0, a_L, a_\nu, \beta_0, h, \|\overline{\nabla}^\perp P_\epsilon\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))}\right).$$

Proof. The distributional time derivative of $\tilde{\nabla}Q_\epsilon(t)$ is defined by the equality

$$\langle\partial_t\tilde{\nabla}Q_\epsilon, \phi\rangle := -\int_0^T \phi'(t)\tilde{\nabla}Q_\epsilon(t)\,dt$$

for all $\phi \in C_c^\infty(0,T)$. To show that $\partial_t\tilde{\nabla}Q_\epsilon(t) \in L^\infty([0,T];(\mathbb{H}\cap H^3(\Omega\times[0,h]))^*))$, we test against functions $v \in \mathbb{H}\cap H^3(\Omega\times[0,h])$. First, recall the definitions of F_ϵ and G_ϵ as the solutions to the linear transport equations with mollified velocity $\overline{\nabla}^\perp P_\epsilon$ as in Step 2. Then we have

$$\begin{aligned}
-\int_0^T \phi'(t)\langle Q_\epsilon(t), v\rangle_{\mathbb{H}}\,dt &= -\int_0^T\int_{\Omega\times[0,h]} \tilde{\nabla}Q_\epsilon(t,x,y,z)\cdot\nabla v(x,y,z)\phi'(t)\,dx\,dy\,dz\,dt \\
&= -\int_0^T B(Q_\epsilon(t), \phi'(t)v)\,dt \\
&= -\int_0^T F(\phi'(t)v)\,dt \\
&= -\int_0^T \Bigg(-\int_{\Omega\times[0,h]} F_\epsilon(t,x,y,z)\phi'(t)v(x)\,dx\,dy\,dz \\
&\quad + \int_{\Omega\times\{0,h\}} \lambda G_\epsilon(t,x,y,z)\phi'(t)v(x,y,z)\,dx\,dy + \int_0^h j_0(z)v|_{\partial\Omega}(z)\phi'(t)\,dz\Bigg)\,dt
\end{aligned} \tag{4.24}$$

Since F_ϵ and G_ϵ are classical solutions to transport equations, we have that

$$\int_0^T \int_{\Omega\times[0,h]} \left(\left(v\phi' + \overline{\nabla}^\perp P_\epsilon \cdot \overline{\nabla} v\phi \right) (F_\epsilon + \beta_0 y) + \phi v a_{L,\epsilon} \right) = 0 \tag{4.25}$$

and

$$\int_0^T \int_{\Omega\times\{0,h\}} \left(\left(v\phi' + \overline{\nabla}^\perp P_\epsilon \cdot \overline{\nabla} v\phi \right) G_\epsilon + \phi v a_{\nu,\epsilon} \right) = 0. \tag{4.26}$$

Plugging (4.25) and (4.26) into (4.24) and noticing that

$$\int_0^T \int_0^h j_0 v\phi' = -\int_0^T \int_0^h (j_0 v)'\phi = 0$$

gives

$$\begin{aligned}
-\int_0^T \phi'(t)\langle Q_\epsilon(t), v\rangle_{\mathbb{H}}\, dt &= -\int_0^T \int_{\Omega\times[0,h]} \left(\left(\overline{\nabla}^\perp P_\epsilon \cdot \overline{\nabla} v\phi \right) (F_\epsilon + \beta_0 y) + \phi v a_{L,\epsilon} \right) \\
&\quad + \int_0^T \int_{\Omega\times\{0,h\}} \lambda \left(\left(\overline{\nabla}^\perp P_\epsilon \cdot \overline{\nabla} v\phi \right) G_\epsilon + \phi v a_{\nu,\epsilon} \right) \\
&\leq \|\overline{\nabla}^\perp P_\epsilon\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} \|\overline{\nabla} v\|_{L^\infty(\Omega\times[0,h])} \|\phi\|_{L^\infty(0,T)} \left(\|F_\epsilon\|_{L^\infty([0,T];L^2(\Omega\times[0,h]))} + \beta_0 h \right) \\
&\quad + \|\phi\|_{L^\infty(0,T)} \|v\|_{L^\infty(\Omega\times[0,h])} \|a_{L,\epsilon}\|_{L^1([0,T];L^2(\Omega\times[0,h]))} \\
&\quad + \Lambda \|\overline{\nabla}^\perp P_\epsilon\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} \|\overline{\nabla} v\|_{L^\infty(\Omega\times\{0,h\})} \|\phi\|_{L^\infty(0,T)} \|G_\epsilon\|_{L^\infty([0,T];L^2(\Omega\times\{0,h\}))} \\
&\quad + \|\phi\|_{L^\infty(0,T)} \|v\|_{L^\infty(\Omega\times\{0,h\})} \|a_{\nu,\epsilon}\|_{L^1([0,T];L^2(\Omega\times[0,h]))} \\
&\leq \|v\|_{H^3(\Omega\times[0,h])} \|\phi\|_{L^\infty(0,T)} \left(1 + \|\overline{\nabla}^\perp P_\epsilon\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} \right) \times \\
&\qquad \left(1 + \|a_L\|_{L^1([0,T];L^2(\Omega\times[0,h]))} + \|a_\nu\|_{L^1([0,T];L^2(\Omega\times[0,h]))} \right) \times \\
&\qquad \left(1 + \|f_0\|_{L^2(\Omega\times[0,h])} + \beta_0 h + \Lambda \|g_0\|_{L^2(\Omega\times\{0,h\})} \right).
\end{aligned}$$

□

Lemma 4.4.3 (Compactness). *Let $\{\epsilon_n\}_{n=1}^\infty$ be a sequence of positive numbers, P_n be a sequence of functions in $C([0,T];\mathbb{H})$, and $S_{\epsilon_n}(P_n) := Q_n$. If there exists M such that the mollified velocity fields $\overline{\nabla}^\perp P_{n,\epsilon_n}$ satisfy*

$$\sup_n \|\overline{\nabla}^\perp P_{n,\epsilon_n}\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} < M$$

then up to a subsequence, there exists $Q \in C([0,T];\mathbb{H})$ such that Q_n converges strongly in $C([0,T];\mathbb{H})$ to Q.

Proof. To set notation, Q_n is the solution to the variational problem

$$B_n(Q_n, v) = F_n(v), \qquad v \in \mathbb{H}$$

described in Step 3. Define the Banach spaces

$$\mathcal{B}_1 = \mathbb{H}^*, \qquad \mathcal{B}_0 = \mathbb{H} \cap H^{\frac{3}{2}}(\Omega \times [0,h]), \qquad \mathcal{B}_2 = \left(\mathbb{H} \cap H^3(\Omega \times [0,h])\right)^*$$

We set $u^* \in \mathbb{H}^*$ as the linear functional on $\mathbb{H}$ defined by $v \to \langle u, v\rangle_{\mathbb{H}}$. This identification provides an isometric linear bijection between $\mathbb{H}$ and $\mathbb{H}^*$. Then by the Rellich-Kondrachov theorem and the observed isomorphism, the embedding of $\mathcal{B}_0$ into $\mathcal{B}_1$ is compact. The inclusion map from $\mathcal{B}_1$ to $\mathcal{B}_2$ is continuous as well. Applying Lemma 4.3.1, invoking the isomorphism between $\mathbb{H}$ and $\mathbb{H}^*$, and using the divergence free property of the mollified transport equations, we have that $Q_n^* \in C([0,T]; \mathbb{H}^*)$, and for $t \in [0,T]$,

$$\|Q^*_{n,\epsilon}(t)\|_{\mathbb{H}^*} \leq C(\Omega, h, \lambda)\left(\|f_0\|_{L^2} + \|g_0\|_{L^2} + \|j_0\|_{L^2} + \|a_L\|_{L^1([0,T];L^2)} + \|a_\nu\|_{L^1([0,T];L^2)}\right). \tag{4.27}$$

In addition, Theorem 4.3.2 provides the bound

$$\|\tilde{\nabla} Q_n(t)\|_{H^{\frac{1}{2}}(\Omega\times[0,h])} \leq C(\Omega, h, \lambda)\left(\|f_0\|_{L^2} + \|g_0\|_{L^2} + \|j_0\|_{L^2} + \|a_L\|_{L^1([0,T];L^2)} + \|a_\nu\|_{L^1([0,T];L^2)}\right), \tag{4.28}$$

showing that $Q_n \in L^\infty([0,T]; \mathcal{B}_0)$. By Lemma 4.4.2 and the existence of the constant M, $\partial_t(Q_n^*)$ is a sequence of operators bounded in $L^\infty([0,T]; \mathcal{B}_2)$, and the assumptions of the Aubins-Lions lemma are satisfied. We have then that Q_n^* is precompact in $C([0,T]; \mathbb{H}^*)$, and thus Q_n is precompact in $C([0,T]; \mathbb{H})$. $\square$

Corollary 4.4.4 (Fixed Points)**.** *Each operator S_ϵ has a fixed point Ψ_ϵ.*

Proof. Lemma 4.4.1 shows that S_ϵ is continuous. By the mollification of the velocity fields, there exists $C(\epsilon)$ such that for all $P \in C([0,T]; \mathbb{H})$,

$$\|\overline{\nabla}^\perp P_\epsilon\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} \leq C(\epsilon)\|P\|_{C([0,T];\mathbb{H})}.$$

Then by Lemma 4.4.3 with $\epsilon_n = \epsilon$ for all n, S_ϵ is a compact operator. By (4.27), the range of S_ϵ is bounded. Therefore, we can apply the Leray-Schauder fixed point theorem (see Evans [48]) to obtain a fixed point Ψ_ϵ. $\square$

4.4.2 Passing to the Limit

Consider the sequence of fixed points Ψ_ϵ to the operators S_ϵ. By definition, $S_\epsilon(\Psi_\epsilon) = \Psi_\epsilon$, and therefore Ψ_ϵ solves the variational problem

$$\begin{aligned} B_{\Psi_\epsilon}(\Psi_\epsilon(t), v) &= \int_{\Omega\times[0,h]} \tilde{\nabla}\Psi_\epsilon(t)\cdot\nabla v \\ &= -\int_{\Omega\times[0,h]} -F_\epsilon(t)v + \int_{\Omega\times\{0,h\}} \lambda G_\epsilon(t)v + \int_0^h v|_{\partial\Omega\times[0,h]} j_0. \\ &= F_{\Psi_\epsilon}(v) \end{aligned}$$

Let us extract a subsequence which we index by $n \in \mathbb{N}$ such that F_{ϵ_n} converges weakly to F in $L^\infty\left([0,T]; L^2(\Omega\times[0,h])\right)$, and G_{ϵ_n} converges weakly to G in $L^\infty([0,T]; L^2(\Omega\times\{0,h\}))$. Define $\Psi(t)$ as the time by time solution to

$$\begin{aligned} B(\Psi(t), v) &= \int_{\Omega\times[0,h]} \tilde{\nabla}\Psi(t)\cdot\nabla v \\ &= -\int_{\Omega\times[0,h]} -F(t)v + \int_{\Omega\times\{0,h\}} \lambda G(t)v + \int_0^h v|_{\partial\Omega\times[0,h]} j_0. \\ &= F(v) \end{aligned}$$

We now pass to the limit to show that Ψ is the solution we seek. As first utilized in [82] and then again in [77], the strong convergence at the level of $\nabla\Psi_{\epsilon_n}$ and the reformulation of the system in terms of $\nabla\Psi_{\epsilon_n}$ give compactness in the nonlinear term of the reformulation. Fix a test function ϕ as in Definition 4.1.2. Then

$$-\int_0^T \int_{\tilde{\Omega}\times[0,h]} \left(\left(\partial_t\phi + \overline{\nabla}^\perp(\Psi_{\epsilon_n} * \eta_{\epsilon_n})\cdot\overline{\nabla}\phi\right) F_{\epsilon_n} + \phi a_{L,n}\right) dx\,dy\,dz\,dt = \int_{\tilde{\Omega}\times[0,h]} \phi|_{t=0} f_{\epsilon_n}\,dx\,dy\,dz \tag{4.29}$$

and

$$\int_0^T \int_{\tilde{\Omega}\times\{0,h\}} \left(\left(\partial_t\phi + \overline{\nabla}^\perp(\Psi_{\epsilon_n} * \eta_{\epsilon_n})\cdot\overline{\nabla}\phi\right) G_{\epsilon_n} + \phi a_{\nu,n}\right) dx\,dy\,dt = -\int_{\tilde{\Omega}\times\{0,h\}} \phi|_{t=0} g_{\epsilon_n}\,dx\,dy \tag{4.30}$$

For each time $t > 0$, let $A_{\epsilon_n}(t) \in \mathbb{H}$ be the solution to

$$\begin{aligned} B_{A,n}(A_{\epsilon_n}(t), v) &:= \int_{\Omega\times[0,h]} \tilde{\nabla} A_{\epsilon_n}(t) \cdot \nabla v \\ &= -\int_{\Omega\times[0,h]} -a_{L,\epsilon}(t) v + \int_{\Omega\times\{0,h\}} a_{\nu,\epsilon}(t) v \\ &= F_{A,n}(v) \end{aligned}$$

Using $\partial_t \phi + \overline{\nabla}^{\perp}(\Psi_{\epsilon_n} * \eta_{\epsilon_n}) \cdot \overline{\nabla}\phi$ and ϕ as test functions in the variational formulations for Ψ_{ϵ_n} and A_{ϵ_n}, respectively, turns (4.29) and (4.30) into

$$\begin{aligned} -\int_0^T \int_{\tilde{\Omega}\times[0,h]} \left(\left(\partial_t \nabla\phi + \overline{\nabla}^{\perp}(\Psi_{\epsilon_n} * \eta_{\epsilon_n}) : \overline{\nabla}\nabla\phi \right) \cdot \tilde{\nabla}\Psi_{\epsilon_n} + \nabla\phi \cdot \tilde{\nabla} A_{\epsilon_n} \right) dx\, dy\, dz\, dt \\ = \int_{\tilde{\Omega}\times[0,h]} \nabla\phi|_{t=0} \cdot \tilde{\nabla}\Psi_{\epsilon_n}|_{t=0}\, dx\, dy\, dz \end{aligned} \tag{4.31}$$

Applying Lemma C.0.1 (refer to Appendix C), we pass to the limit to obtain

$$\begin{aligned} -\int_0^T \int_{\tilde{\Omega}\times[0,h]} \left(\left(\partial_t \nabla\phi + \overline{\nabla}^{\perp}\Psi : \overline{\nabla}\nabla\phi \right) \cdot \tilde{\nabla}\Psi + \nabla\phi \cdot \tilde{\nabla} A \right) dx\, dy\, dz\, dt \\ = \int_{\tilde{\Omega}\times[0,h]} \nabla\phi|_{t=0} \cdot \tilde{\nabla}\Psi|_{t=0}\, dx\, dy\, dz \end{aligned}$$

Rearranging the variational formulation now for Ψ gives

$$-\int_0^T \int_{\tilde{\Omega}\times[0,h]} \left(\left(\partial_t \phi + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\phi \right) F + \phi a_L \right) dx\, dy\, dz\, dt = \int_{\tilde{\Omega}\times[0,h]} \phi|_{t=0} f\, dx\, dy\, dz$$

and

$$\int_0^T \int_{\tilde{\Omega}\times\{0,h\}} \left(\left(\partial_t \phi + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla}\phi \right) G + \phi a_\nu \right) dx\, dy\, dt = -\int_{\tilde{\Omega}\times\{0,h\}} \phi|_{t=0} g\, dx\, dy.$$

The final part of the proof consists of showing that $\Psi(t)$ solves a variational problem for all $t \in [0,T]$ which verifies the compatibility condition Definition 4.1.1. By construction of the approximate solution operators, $\Psi_{\epsilon_n}(t)$ solves the variational problem with data

$$\left(F_{\epsilon_n}(t)|_{\Omega\times[0,h]}, G_{\epsilon_n}(t)|_{\Omega\times\{0,h\}}, j_0 \right).$$

In addition, $F_{\epsilon_n}(t)$ and $G_{\epsilon_n}(t)$ are supported in a neighborhood of order ϵ_n around Ω for all time $t \in [0,T]$. Then using the weak convergence of $F_n|_{\Omega\times[0,h]}$ to F and $G_n|_{\Omega\times\{0,h\}}$ to G, we have

$$\int_{\Omega\times[0,h]} F_n(t) + \int_{\Omega\times\{0,h\}} G_n(t) + \int_0^h j_0 \to$$
$$\int_{\Omega\times[0,h]} f_0 + \int_0^t \int_{\Omega\times[0,h]} a_L + \int_{\Omega\times\{0,h\}} g_0 + \int_0^t \int_{\Omega\times\{0,h\}} a_\nu + \int_0^h j_0$$

Using the assumption that (f_0, g_0, j_0) and (a_L, a_ν) satisfy Definition 4.1.1 shows that $\Psi(t)$ solves an elliptic problem with compatible data. Then by Lemma 4.3.1, $\mathcal{L}(\Psi) = F$ and $\partial_\nu \Psi = G$ in the traditional weak sense.

We have thus shown that Ψ satisfies part (4) of Theorem 3.1, and therefore Definition 4.1.2 and part (1) of Theorem 3.1. For part (2), the choice of Ψ as a weak limit of functions belonging to $L^\infty([0,T];\mathbb{H})$ implies that $\Psi(t) \in \mathbb{H}$ for almost every t. Therefore, Ψ must depend only on z on the lateral boundary, and there exists $c(t,z)$ such that $\Psi|_{\partial\Omega\times[0,h]} = c(t,z)$ for almost every time. To show part (3), first note that in light of the $H^{\frac{1}{2}}(\Omega\times[0,h])$ bound on $\nabla\Psi$, $\overline{\nabla}\Psi\cdot\nu_s(t)$ is well-defined in $L^2(\partial\Omega\times[0,h])$ for almost every time. Assuming now that $j_0 \in H^{\frac{1}{2}}(0,h)$, let $\alpha_n(z)$ be a compactly supported smooth function in $(\frac{1}{n}, h-\frac{1}{n})$ such that $\alpha_n(z) = 1$ for all $z \in (\frac{2}{n}, h-\frac{2}{n})$. Applying Theorem 4.3.2 to $\alpha_n\Psi(t)$ shows that $\alpha_n\Psi(t) \in H^2(\Omega\times[0,h])$, and therefore $\overline{\Delta}\Psi(t,z) \in L^2(\Omega)$ for $z \in (\frac{2}{n}, h-\frac{2}{n})$. Then

$$\int_{\Omega\times\{z\}} \overline{\Delta}\Psi = \int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi\cdot\nu_s$$

and applying Lemma 4.3.1 shows part (3). Finally, the bounds in part (5) follow from the divergence free property of the flow and Theorem 4.3.2, completing the proof of the theorem.

Chapter 5

Nonuniqueness of Weak Solutions

5.1 Overview

In this chapter, we pose the inviscid 3D QG equations for $(t,x,y,z) \in \mathbb{R} \times \mathbb{T}^3$.

$$\begin{cases} \partial_t(\Delta\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \nabla(\Delta\Psi) = 0 & (t,x,y,z) \in \mathbb{R} \times \mathbb{T}^3 \\ \partial_t(\partial_\nu\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \nabla(\partial_\nu\Psi) = 0 & (t,x,y,z) \in \mathbb{R} \times \mathbb{T}^2 \times \{0, 2\pi\}. \end{cases}$$

We shall exclusively use the reformulation due to Puel and Vasseur [82].

$$\begin{cases} \partial_t(\nabla\Psi) + \overline{\nabla}^{\perp}\Psi \cdot \nabla(\nabla\Psi) = \operatorname{curl}(Q) & (t,x,y,z) \in \mathbb{R} \times \mathbb{T}^3 \\ \operatorname{curl}(Q) \cdot (0,0,1) = 0 & (t,x,y,z) \in \mathbb{R} \times \mathbb{T}^2 \times \{0, 2\pi\}. \end{cases}$$

Recall from Chapter 3 that weak solutions to the reformulated problem are defined via the following equality for all test functions ϕ in $C^\infty(\mathbb{R} \times \mathbb{T}^3)$ which are compactly supported in time:

$$\int_{\mathbb{R}} \int_{\mathbb{T}^3} \partial_t(\nabla\phi) \cdot \nabla\Psi + \nabla\Psi \cdot \left(\overline{\nabla}^{\perp}\Psi \cdot \nabla\nabla\phi \right) \, dt \, dx \, dy \, dz = 0.$$

Furthermore, we showed in Chapter 3 that under sufficient integrability assumptions on $\Delta\Psi$ and $\partial_\nu\Psi$ (not satisfied by the solutions we construct in this chapter), weak solutions to the reformulated problem are weak solutions to the original system of equations, and vice versa. The purpose of this chapter is to investigate the rigidity/flexibility and demonstrate the non-uniqueness of such weak solutions to 3D QG.

Theorem 5.1.1. *Let $e : \mathbb{R} \to [0, \infty)$ be a smooth, compactly supported function and $\zeta \in \left(0, \frac{1}{5}\right)$. Then there exist vector fields $\nabla\Psi \in C^\zeta(\mathbb{R} \times \mathbb{T}^3)$ and $Q \in L^\infty\left(\mathbb{R}; C^{2\zeta}(\mathbb{T}^3)\right)$ such that $\nabla\Psi$ is a weak solution to 3D QG and*

$$\int_{\mathbb{T}^3} |\nabla\Psi(t,x,y,z)|^2 \, dx \, dy \, dz = e(t).$$

The proof of Theorem 5.1.1 proceeds via a convex integration scheme. While we shall postpone a more detailed description of the proof for the time being, we emphasize that the stratification of the velocity field plays a key role. The stratification provides a link between 3D QG and the two-dimensional Euler equations. We therefore obtain the following theorem as a corollary of our construction.

Theorem 5.1.2. *Consider the two-dimensional Euler equations*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 & (t,x,y) \in \mathbb{R} \times \mathbb{T}^2 \\ \nabla \cdot u = 0 & (t,x,y) \in \mathbb{R} \times \mathbb{T}^2. \end{cases}$$

Given a smooth, compactly supported energy profile $e : \mathbb{R} \to [0,\infty)$ and $\zeta \in \left(0, \frac{1}{5}\right)$, there exists (u,p) which solves the equations in the sense of distributions with $u \in C^\zeta(\mathbb{R} \times \mathbb{T}^2)$, $p \in L^\infty\left(\mathbb{R}; C^{2\zeta}(\mathbb{T}^3)\right)$, and

$$\int_{\mathbb{T}^2} |u(t,x,y)|^2 \, dx \, dy = e(t).$$

Theorem 5.1.2 is not a new result. Convex integration for 2D Euler equations was first considered by Choffrut, De Lellis and Székelyhidi in the class of continuous solutions [23]. In [9], Buckmaster, Shkoller, and Vicol observe that by replacing the Beltrami waves used in [12] with Beltrami plane waves, non-uniqueness for 2D Euler can be shown in the class C^ζ for $\zeta \in \left(0, \frac{1}{5}\right)$ following the methods outlined in their paper. Nonetheless, we include an explicit proof of Theorem 5.1.2 since it follows simply from our construction.

In order to comment on the sharpness of our results and the commonalities and contrasts with other convex integration schemes, let us describe the main aspects of the proof. We build a solution $\nabla\Psi$ through an iterative process which specifies the Littlewood-Paley projections of $\nabla\Psi$ at each stage. After q stages of this iteration, we have vector fields $\nabla\Psi_q$, $\mathrm{curl}(Q_q)$ and E_q which solve

$$\partial_t(\nabla\Psi_q) + \overline{\nabla} \cdot \left(\nabla\Psi_q \otimes \overline{\nabla}^\perp \Psi_q\right) = \mathrm{curl}(Q_q) + E_q.$$

Throughout, we identify $\nabla\Psi_q \otimes \overline{\nabla}^\perp \Psi_q$ with a matrix whose rows are specified by the components of $\nabla\Psi_q$ and whose columns are specified by the components of $\overline{\nabla}^\perp \Psi_q$. Differential operators with a bar such as $\overline{\nabla}\cdot$ include derivatives in x and y only. For example, the divergence $\overline{\nabla}\cdot$ of the above matrix is taken row by row and differentiates in x and y only, thus ignoring the third column (which is already zero). At this stage, each function is supported in frequency in a ball of radius λ_q around the origin. The goal is to send E_q to 0 as $q \to \infty$, thus obtaining a solution to 3D QG in the limit. In order to minimize E_q, we shall add the next Littlewood-Paley projection of $\nabla \Psi_q$, which we call $\nabla \mathbb{W}_{q+1}$, in the hopes of making

$$\overline{\nabla} \cdot \left(\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1} \right) - E_q \approx 0 \tag{5.1}$$

at low frequencies. In order to facilitate this cancellation, we first require an "inverse divergence" operator $\mathcal{D}$ satisfying

$$\overline{\nabla} \cdot \mathring{M}_q = \overline{\nabla} \cdot \left(\mathcal{D}(E_q) \right).$$

In order for (5.1) to hold, $\mathcal{D}$ must output a matrix field $\mathring{M}_q$ which resembles a tensor product $\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1}$. Therefore, we must define $\mathcal{D}$ to output matrices which have zeroes in the third row. In addition, considering that the divergence $\overline{\nabla} \cdot$ is in x and y only, it is natural for $\mathcal{D}$ to be a convolution operator in x and y only as well. After constructing such a $\mathcal{D}$ (see Proposition 5.3.3), it is clear that the amount of regularity it gains will depend on only the first two components of the frequency modes of E_q. We will refer to these modes throughout the paper as the "x and y frequency modes." This also serves as the first indication that a successful convex integration scheme for 3D QG can also produce solutions to 2D Euler. Thus we have encountered the first distinctive aspect of our argument:

The inverse divergence only gains regularity in x and y, and so we must choose frequency modes for $\nabla \Psi_q$ which avoid the z-axis.

Let us describe some important characteristics of the perturbation $\nabla \mathbb{W}_{q+1}$. The size of $\mathring{M}_q$ is quantified by the parameter δ_{q+1}, and therefore it is natural for the amplitude of $\nabla \mathbb{W}_{q+1}$ to be roughly $\delta_{q+1}^{\frac{1}{2}}$. In addition, we specify $\nabla \mathbb{W}_{q+1}$ to exist in frequency in a sphere of radius $\lambda_{q+1} > \lambda_q$ so as not to interfere too drastically with the rest of the terms in the equation. However, in order for $\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1}$ to help eliminate lower frequency errors, it must contain a low frequency portion. That is, if $\nabla \mathbb{W}_{q+1}$ has a non-zero coefficient c_k at a frequency mode $\lambda_{q+1} k$ for some $k \in \mathbb{S}^2$, it should have a corresponding coefficient $\overline{c_k}$ at the mode $-\lambda_{q+1} k$. The perturbations in convex integration schemes are generally constructed using stationary solutions of the underlying equation, and thus we seek $\nabla \mathbb{W}_{q+1}$ which approximately solves

$$\begin{cases} \overline{\nabla} \cdot \left(\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1} \right) = \operatorname{curl}(Q) & (x,y,z) \in \mathbb{T}^3 \\ \operatorname{curl}(Q) \cdot (0,0,1) = 0 & (x,y,z) \in \mathbb{T}^2 \times \{0, 2\pi\}. \end{cases}$$

A straightforward calculation (Lemma 5.3.5) shows that eigenfunctions of the Laplacian will satisfy the first equation. The second equation, however, requires specific behavior of the eigenfunctions at the boundary, highlighting another distinguishing aspect of 3D QG.

The dynamics of 3D QG at the boundaries of $\mathbb{T}^3$ requires the use of stationary solutions $\nabla \mathbb{W}_{q+1}$ such that $\overline{\nabla}^{\perp} \mathbb{W}_{q+1} \cdot \overline{\nabla}(\partial_z \mathbb{W}_{q+1})$ vanishes at $z = 0, 2\pi$.

A natural way to achieve this would be to impose that $\partial_z \mathbb{W}_{q+1}$ vanishes at $z = 0, 2\pi$. Therefore, if the Fourier series of $\mathbb{W}_{q+1}$ contains the term $c_k e^{i(k_1,k_2,k_3)\cdot(x,y,z)}$, it should also contain the term $c_k e^{i(k_1,k_2,-k_3)\cdot(x,y,z)}$. Then

$$\partial_z \mathbb{W}_{q+1} = c_k e^{i(k_1,k_2,0)\cdot(x,y,0)} i k_3 \left(e^{ik_3 z} - e^{-ik_3 \cdot z} \right)$$

will vanish at $z = 0, 2\pi$. However, this has the unfortunate effect of annihilating the low frequency portion of $\overline{\nabla}^{\perp} \mathbb{W}_{q+1} \partial_z \mathbb{W}_{q+1}$ in the entirety of $\mathbb{T}^3$. Indeed, choosing modes

$$(k_1, k_2, k_3), \quad (-k_1, -k_2, -k_3), \quad (k_1, k_2, -k_3), \quad (-k_1, -k_2, k_3),$$

denoting $\bar{k}^{\perp} = (-k_2, k_1, 0)$, and writing out the low frequency portion of the third row of $\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1}$, we obtain

$$|c_k|^2 (k_3) \bar{k}^{\perp} + |c_k|^2 (-k_3) \bar{k}^{\perp} = 0.$$

Towards the goal of producing stationary solutions, we instead introduce a cutoff function L_{q+1} which depends on z only and define our perturbation as $\nabla \left(\mathbb{W}_{q+1} L_{q+1} \right)$. The viability of the cutoff function is visible in the equality

$$\begin{aligned} \overline{\nabla} \cdot \left(\nabla \left(L_{q+1} \mathbb{W}_{q+1} \right) \otimes \overline{\nabla}^{\perp} \left(L_{q+1} \mathbb{W}_{q+1} \right) \right) &= L_{q+1}^2 \overline{\nabla} \cdot \left(\nabla \mathbb{W}_{q+1} \otimes \overline{\nabla}^{\perp} \mathbb{W}_{q+1} \right) \\ &= L_{q+1}^2 \operatorname{curl}(Q) \\ &= \operatorname{curl}(L_{q+1}^2 Q) - \text{lower order terms} \end{aligned}$$

We prove and discuss this equality in Lemma 5.3.5 and Eq. (5.17), with the basic idea being that we have constructed solutions *which are stationary to leading order.* We couple this equality with an additional inductive assumption (see (5.7)); namely, that the spatial supports of $\nabla\Psi_q$, $\mathrm{curl}(Q_q)$, and E_q are contained in the region where $L_{q+1} = 1$. Since the inverse divergence is a convolution in x and y only, $\mathring{M}_q = \mathcal{D}(E_q)$ will only be supported in the region where $L_{q+1} \equiv 1$ as well. Furthermore, the advection operator $D_{t,q} := \partial_t + \overline{\nabla}^\perp\Psi_q \cdot \overline{\nabla}$ applied to L_{q+1} satisfies

$$\partial_t L_{q+1} + \overline{\nabla}^\perp\Psi_q \cdot \overline{\nabla} L_{q+1} = 0.$$

Thus, multiplication by L_{q+1} *commutes* with the important operators in our scheme and does not interfere with the oscillatory term, making its implementation rather simple.

5.1.1 Connection to 2D Euler

Suppose that one were to construct a solution to 3D QG which did not depend on z. While such a solution would then ignore all the important physical aspects of three-dimensional quasi-geostrophic dynamics, under this condition the equation becomes

$$\partial_t\left(\overline{\nabla}\Psi\right) + \overline{\nabla}^\perp\Psi \cdot \overline{\nabla}\left(\overline{\nabla}\Psi\right) = \overline{\nabla}^\perp Q,$$

which after setting $u = \overline{\nabla}^\perp\Psi$ and $p = Q$ becomes 2D Euler. To construct solutions to 2D Euler using our scheme, we simply lift all restrictions on the spatial support, discard the localizer L_{q+1}, and choose frequency modes with vanishing third component at each stage of the iteration so that $\partial_z\Psi \equiv 0$. Thus, it is natural that our solution should produce Hölder continuous solutions in classes C^ζ for $\zeta \in \left(0, \frac{1}{5}\right)$, as the Onsager conjecture for 2D Euler remains open in between $\frac{1}{5}$ and $\frac{1}{3}$.

5.1.2 An Onsager Conjecture for 3D QG

We will show in this chapter that weak solutions to 3D QG conserve the energy $\|\nabla\Psi(t)\|_{L^2}^2$ when $\nabla\Psi$ belongs to the space $L^\infty_{t,z}\left(\mathring{B}^s_{3,\infty}\left(\mathbb{R}^2_{x,y}\right)\right)$ for $s > \frac{1}{3}$. The stratification of the velocity field allows for the lower regularity in the z variable. Essentially, one only needs to integrate by parts in x and y to show that the energy flux cannot contribute to the spontaneous production or dissipation of energy. This leads us to conjecture the following dichotomy concerning the flexibility of weak solutions to 3D QG:

For any $\zeta \in \left(0, \frac{1}{3}\right)$, there exists infinitely many weak solutions which do not conserve the energy $\|\nabla\Psi(t)\|_{L^2}$. In Hölder classes above $\frac{1}{3}$, the energy of a weak solution is constant in time.

Therefore, Theorem 5.1.1 does not close the appropriately formulated version of the Onsager conjecture for 3D QG.

5.1.3 Relation of Our Result to Non-Uniqueness for 2D SQG

The Onsager threshold for the inviscid SQG equation is conjectured to correspond to $\partial_z\Psi \in L^\infty$ and is not fully resolved yet (see [9] for a thorough discussion). As our solutions vanish at $z = 0$ and $z = 2\pi$, Theorem 5.1.1 does not imply any results for 2D SQG. Nor does our result follow from the non-uniqueness of 2D SQG shown in [9]. In 2D SQG, one has that $\Delta\Psi(t) \equiv 0$ for all time t. Physically, this represents an atmosphere which is at rest in the interior, and in which all the dynamics occur at the boundary. However, for 3D QG, one does not rule out the possibility of interior vorticity, allowing for the addition of high frequency oscillations not only at the boundary, but in the interior as well. The solutions we construct are not harmonic. Therefore, it is natural that they should be more regular than the dissipative solutions to 2D SQG. Perhaps coupling the techniques from [9] and the present paper while finding more flexible building blocks could provide progress on reaching the threshold $C^{\frac{1}{3}}$ for 3D QG.

5.1.4 History of Convex Integration

Non-uniqueness of weak solutions to the Euler equations has been known for some time, with proofs given by Scheffer [85] and Shnirelman [87]. The modern convex integration techniques were developed by De Lellis and Székelyhidi in [68], [41], and [67]. After a number of results investigating the flexibility of solutions and obtaining partial progress towards Onsager's conjecture for the 3D Euler equations (cf. [25], [24], [11], [12], [13]), [39],

[67], [57], [55]), a proof of the full conjecture was given by Isett [56]. In a subsequent work, Buckmaster, De Lellis, Székelyhidi, and Vicol [14] treat the case of dissipative solutions in the full Onsager regime. In [58], Isett constructed Hölder continuous solutions obeying the local energy inequality. Non-uniqueness for 2D SQG (see the below section on quasi-geostrophic flows) was shown by Buckmaster, Shkoller, and Vicol [9]. In addition, non-uniqueness of 3D Navier-Stokes has been demonstrated by Buckmaster and Vicol [10], and Buckmaster, Colombo, and Vicol [8]. Stationary solutions to the 4D Navier-Stokes equations have been constructed by Luo [70], with a different construction for 3D offered by Luo and Cheskidov [22].

5.2 Conservation of Energy

In this section we prove the following theorem following the proof in [77]. For this reason, the proof is stated for $(QG)_I$ posed in the upper half space. However, elementary modifications can be made to show that the same result holds for $(QG)_I$ posed in $\mathbb{T}^3$.

Theorem 5.2.1. *Let $\nabla\Psi$ be a weak solution to the reformulated problem with no forcing such that*

$$\nabla\Psi \in C\left([0,T); L^2(\mathbb{R}^3_+)\right) \cap L^3\left([0,T)\times[0,\infty); \mathring{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)\right)$$

for some $\alpha > \frac{1}{3}$. Then $\|\nabla\Psi(t)\|_{L^2(\mathbb{R}^3_+)} = \|\nabla\Psi_0\|_{L^2(\mathbb{R}^3_+)}$ for $t \in [0,T)$.

Proof. Define for all time

$$\begin{aligned}(\nabla\Psi^\epsilon)^\epsilon(z,x) &:= (\nabla\Psi(z,\cdot) * \gamma_\epsilon) * \gamma_\epsilon(x)\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \nabla\Psi(z, x - x' - \bar{x})\gamma_\epsilon(x')\gamma_\epsilon(\bar{x})\, dx'\, d\bar{x}\end{aligned}$$

that is, we convolve $\nabla\Psi$ twice with a mollifier γ_ϵ in x only, z by z. The extra mollification is for passage onto the nonlinear term later. Strictly speaking, $(\nabla\Psi^\epsilon)^\epsilon$ is not an admissible test function; it lacks compact support in space and time, and differentiability in z and t. However, let us proceed formally for the time being, and assume that $(\nabla\Psi^\epsilon)^\epsilon$ is admissible and that $\nabla\Psi$ is differentiable in time. Multiplying (rQG) by $(\nabla\Psi^\epsilon)^\epsilon$ and integrating in space and from time 0 to t, we obtain

$$\begin{aligned} E_\epsilon(t) - E_\epsilon(0) &:= \int_0^\infty \int_{\mathbb{R}^2} \nabla\Psi^\epsilon(t)\cdot\nabla\Psi^\epsilon(t)\,dx\,dz - \int_0^\infty \int_{\mathbb{R}^2} \nabla\Psi^\epsilon(0)\cdot\nabla\Psi^\epsilon(0)\,dx\,dz \\ &= -2\int_0^t \int_0^\infty \int_{\mathbb{R}^2} \left(\overline{\nabla}^\perp\Psi : \overline{\nabla}(\nabla\Psi^\epsilon)^\epsilon\right)\cdot\nabla\Psi\,dx\,dz\,d\tau \end{aligned} \tag{5.2}$$

We can now apply Proposition D.0.1(1 in Appendix D) to the right hand side to move the mollifier over, introduce the commutator between multiplication and mollification, and rewrite the nonlinear terms using tensor notation, obtaining

$$\begin{aligned} E_\epsilon(t) - E_\epsilon(0) = -2\int_0^t \int_0^\infty \int_{\mathbb{R}^2} &\left\langle \left(\overline{\nabla}^\perp\Psi\otimes\nabla\Psi\right)^\epsilon - \left(\overline{\nabla}^\perp\Psi^\epsilon\otimes\nabla\Psi^\epsilon\right), \overline{\nabla}\nabla\Psi^\epsilon\right\rangle\,dx\,dz\,d\tau \\ - 2\int_0^t \int_0^\infty \int_{\mathbb{R}^2} &\left\langle \overline{\nabla}^\perp\Psi^\epsilon\otimes\nabla\Psi^\epsilon, \overline{\nabla}\nabla\Psi^\epsilon\right\rangle\,dx\,dz\,d\tau \end{aligned}$$

Integrating by parts in x for fixed z and τ gives that the second term is equal to zero. Applying Proposition D.0.1(2) z by z with $f = \overline{\nabla}^\perp\Psi$ and $g = \nabla\Psi$ to the first term, we have

$$\begin{aligned} E_\epsilon(t) - E_\epsilon(0) = -2\int_0^t \int_0^\infty \iiint_{(\mathbb{R}^2)^3} &\Big\langle \left(\overline{\nabla}^\perp\Psi(x-\bar{x}) - \overline{\nabla}^\perp\Psi(x)\right)\otimes \\ &\left(\nabla\Psi(x-\bar{x}) - \nabla\Psi(x-x')\right), \overline{\nabla}\nabla\Psi(x)\Big\rangle\cdot\gamma_\epsilon(\bar{x})\gamma_\epsilon(x')\,d\bar{x}\,dx'\,dx\,dz\,d\tau. \end{aligned}$$

Now apply Hölder's inequality in x to obtain

$$\begin{aligned} |E_\epsilon(t) - E_\epsilon(0)| \le C\int_0^t \int_0^\infty \iint_{(\mathbb{R}^2)^2} &\|\overline{\nabla}^\perp\Psi(z,\tau,\cdot-\bar{x}) - \overline{\nabla}^\perp\Psi(z,\tau,\cdot)\|_{L^3(\mathbb{R}^2)}\gamma_\epsilon(\bar{x})\gamma_\epsilon(x') \\ &\times\|\nabla\Psi(z,\tau,\cdot) - \nabla\Psi(z,\tau,\cdot-(x'-\bar{x}))\|_{L^3(\mathbb{R}^2)}\|\overline{\nabla}\nabla\Psi(z,\tau,\cdot)\|_{L^3(\mathbb{R}^2)}\,d\bar{x}\,dx'\,dz\,d\tau. \end{aligned}$$

Integrating in $\bar{x}$ and x', using the fact that γ_ϵ has integral one, and applying Proposition D.0.1(3) z by z for $\bar{x}, x'-\bar{x} < C\epsilon$ gives

$$\begin{aligned} |E_\epsilon(t) - E_\epsilon(0)| &\le C\int_0^t \int_0^\infty \|\overline{\nabla}^\perp\Psi(z,\tau,\cdot)\|_{\mathring{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)}\|\nabla\Psi(z,\tau,\cdot)\|^2_{\mathring{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)}\epsilon^{3\alpha-1}\,dz\,d\tau \\ &\le C\epsilon^{3\alpha-1}\|\nabla\Psi\|^3_{L^3\left([0,T)\times[0,\infty);\mathring{B}^\alpha_{\infty,\infty}(\mathbb{R}^2)\right)} \end{aligned}$$

which approaches 0 as $\epsilon\to 0$ if $\alpha > \frac{1}{3}$.

We must now account for that fact that $(\nabla\Psi^\epsilon)^\epsilon$ is not an admissible test function. Replacing Ψ (which is a well-defined function in $C\left([0,T);L^6(\mathbb{R}^3_+)\right)$ by Sobolev embedding) with

$$\Psi_\eta := \left(\mathcal{X}_{\{|(x,z)|\leq\frac{1}{\eta},\tau\leq T-\eta\}}\Psi\right) * \Gamma_\eta$$

for Γ_η a space-time mollifier in $\mathbb{R}^4$ ensures compact support and differentiability in z and t. Then after mollifying as before in x, we can use $(\nabla(\Psi_\eta)^\epsilon)^\epsilon$ as a test function. It is well known that (5.2) holds when differentiability in time is replaced with $C\left([0,T);L^2(\mathbb{R}^3_+)\right)$. Passing to the limit in η first and then in ϵ gives that

$$\|\nabla\Psi(t)\|^2_{L^2(\mathbb{R}^3_+)} - \|\nabla\Psi(0)\|^2_{L^2(\mathbb{R}^3_+)} = \lim_{\epsilon\to 0} E_\epsilon(t) - E_\epsilon(0) = 0,$$

completing the proof. $\square$

5.3 Preliminaries for Convex Integration

We start by defining our inverse divergence operator $\mathcal{D}$, which will be a convolution kernel in x and y only. We will do this in several steps, the first of which is as follows.

Proposition 5.3.1 (Inverse Divergence of $\overline{\nabla}$). *Let $\overline{\nabla}f : \mathbb{T}^3 \to \mathbb{R}^3$ have zero mean on $\mathbb{T}^3$. Define $\mathcal{E}(\overline{\nabla}f)$ by*

$$\mathcal{E}(\overline{\nabla}f) = \begin{bmatrix} \partial_{22}(-\overline{\Delta})^{-1}f - \partial_{11}(-\overline{\Delta})^{-1}f & -2\partial_{12}(-\overline{\Delta})^{-1}f \\ -2\partial_{12}(-\overline{\Delta})^{-1}f & \partial_{11}(-\overline{\Delta})^{-1}f - \partial_{22}(-\overline{\Delta})^{-1}f \end{bmatrix}$$

Then $\overline{\nabla}\cdot\mathcal{E}(\overline{\nabla}f) = \overline{\nabla}f$, and $\mathcal{E}(\overline{\nabla}f)$ is symmetric and traceless. If $\operatorname{supp}\hat{f} \subset \left\{|\overline{k}| \geq \lambda\right\}$, *then*

$$\left\|\mathcal{E}(\overline{\nabla}f)\right\|_{C^0} \lesssim \frac{1}{\lambda}\|\overline{\nabla}f\|_{C^0}.$$

Proof. The equality of $\overline{\nabla}\cdot\mathcal{E}(\overline{\nabla}f)$ and $\overline{\nabla}f$ proceeds by direct computation. The estimate on the C^0 norm follows from Lemma D.0.2 and the fact that the multiplier is homogeneous of degree -1. Notice that $\mathcal{E}$ is identical to the inverse divergence operator defined in [9] after switching the rows and changing the sign of the new second row. $\square$

The second step in defining our inverse divergence operator is the following.

Proposition 5.3.2 (Inverse Divergence of Scalar Functions). *Let $g : \mathbb{T}^3 \to \mathbb{R}$ have zero mean on $\mathbb{T}^3$. Define $\mathcal{I}(g)$ by*

$$\mathcal{I}(g) = -(-\overline{\Delta})^{-1}\overline{\nabla} g.$$

Then $\overline{\nabla} \cdot \mathcal{I}(g) = g$. If $\operatorname{supp} \hat{g} \subset \left\{|\bar{k}| \geq \lambda\right\}$, then

$$\|\mathcal{I}(g)\|_{C^0} \lesssim \frac{1}{\lambda}\|g\|_{C^0}.$$

Proof. As before, the equality proceeds by direct computation and the estimate is a corollary of Lemma D.0.2 and the homogeneity of the symbol. $\square$

The inverse divergence we use will be applied to vector fields for which the first two components are the gradient of a scalar-valued function, while the third component is a (different) scalar-valued function.

Proposition 5.3.3 (Inverse Divergence of $\mathbb{X} := (\partial_x f, \partial_y f, g)$). *Let $\mathbb{X} = (\overline{\nabla} f, g) : \mathbb{T}^3 \to \mathbb{R}^3$ have zero mean on $\mathbb{T}^3$. Define $\mathcal{D}(\mathbb{X})$ to be the 3×3 matrix*

$$\mathcal{D}(\mathbb{X}) = (-\overline{\Delta})^{-1} \begin{bmatrix} (\partial_{22} - \partial_{11})(f) & -2\partial_{12}(f) & 0 \\ -2\partial_{12}(f) & (\partial_{11} - \partial_{22})(f) & 0 \\ -\partial_1 g & -\partial_2 g & 0 \end{bmatrix}$$

Then $\overline{\nabla} \cdot \mathcal{D}(\mathbb{X}) = \mathbb{X}$. If $\operatorname{supp} \hat{\mathbb{X}} \subset \left\{|\bar{k}| \geq \lambda\right\} \times \mathbb{Z}$, then

$$\|\mathcal{D}(\mathbb{X})\|_{C^0} \lesssim \frac{1}{\lambda}\|\mathbb{X}\|_{C^0}.$$

Proof. The equality of $\overline{\nabla} \cdot \mathcal{D}\mathbb{X}$ and $\mathbb{X}$ proceeds by direct computation. To prove the estimate, first apply Lemma D.0.3 to see that $\overline{\mathbb{P}}_{\geq\lambda}(\mathbb{X}) = \mathbb{X}$. Then using Bernstein's inequality in x and y and the fact that $\mathcal{D}$ is a convolution operator in x and y only gives the claim. $\square$

The following lemma is the analogue of the so-called geometric lemma from [9] and describes the mechanism by which we can cancel out errors with the addition of high-frequency waves.

Lemma 5.3.4 (Choosing Frequency Modes). *Define*

$$\mathcal{M} = \left\{ \begin{bmatrix} m_1 & m_2 & 0 \\ m_3 & -m_1 & 0 \\ m_4 & m_5 & 0 \end{bmatrix} : m_i \in \mathbb{R} \right\}$$

Then there exist matrices $M_1, M_2 \in \mathcal{M}$, $\epsilon > 0$, disjoint finite subsets $\Omega_j \in \mathbb{Q}^3 \cap \mathbb{S}^2$ for $j = 1, 2$, and smooth positive functions defined in a neighborhood of M_j and indexed by $k \in \Omega_j$ which we call $c_{j,k} \in C^\infty(B_{\epsilon,\mathcal{M}}(M_j))$ such that

1. *Both of the sets Ω_j are at positive distance from the z-axis*
2. *$\Omega_j = -\Omega_j$ and $c_{j,k} = c_{j,-k}$*
3. *$13\Omega_j \in \mathbb{Z}^3$ for $j = 1, 2$*
4. *For $j = 1, 2$ and $\forall M \in B_\epsilon(M_j)$, we have*

$$M = \frac{1}{2} \sum_{k \in \Omega_j} (c_{j,k}(M))^2 k \otimes \bar{k}^\perp.$$

5. *Furthermore, if $M = M_j + N$ where $N \in \mathcal{M}$ satisfies $N^{12} = N^{21}$ (i.e., the top left block of N is symmetric in addition to being traceless), then*

$$\sum_{k \in \Omega_j} (c_{j,k}(M))^2 = 1. \tag{5.3}$$

Proof. We begin by constructing Ω_1^+, where Ω_1^- will be defined as $-\Omega_1^+$ and $\Omega = \Omega_1^+ \cup \Omega_1^-$. We choose the following vectors (inspired by the fact that $(5, 12, 13)$ and $(3, 4, 5)$ are Pythagorean triples):

$$k_1 = \frac{1}{13}(5, 0, 12), \quad k_2 = \frac{1}{13}(3, 4, -12), \quad k_3 = \frac{1}{13}(3, -4, 12),$$
$$k_4 = \frac{1}{13}(0, 5, 12) \quad k_5 = \frac{1}{13}(3, 4, 12).$$

Then it is clear that (1) and (3) hold for Ω_1^+. Constructing the corresponding matrices $k_i \otimes \bar{k}_i^\perp$, denoted m_{k_i}, we have

$$m_{k_1} = \frac{1}{169} \begin{bmatrix} 0 & 25 & 0 \\ 0 & 0 & 0 \\ 0 & 60 & 0 \end{bmatrix}, \quad m_{k_2} = \frac{1}{169} \begin{bmatrix} -12 & 9 & 0 \\ -16 & 12 & 0 \\ 48 & -36 & 0 \end{bmatrix}, \quad m_{k_3} = \frac{1}{169} \begin{bmatrix} 12 & 9 & 0 \\ -16 & -12 & 0 \\ 48 & 36 & 0 \end{bmatrix}$$

$$m_{k_4} = \frac{1}{169}\begin{bmatrix} 0 & 0 & 0 \\ -25 & 0 & 0 \\ -60 & 0 & 0 \end{bmatrix}, \quad m_{k_5} = \frac{1}{169}\begin{bmatrix} -12 & 9 & 0 \\ -16 & 12 & 0 \\ -48 & 36 & 0 \end{bmatrix}$$

Furthermore, one can check that the set $\{m_{k_i}\}$ is a linearly independent set within $\mathcal{M}$. After identifying $\mathcal{M}$ with $\mathbb{R}^5$, define the function $f_1 : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$f_1(x, y, z, s, t) = xm_{k_1} + ym_{k_2} + zm_{k_3} + sm_{k_4} + tm_{k_5}.$$

Then $f_1 \in C^\infty$ and $Df_1|_{\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)}$ is invertible. Define $M_1 := f_1\left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$. Applying the inverse function theorem, we obtain ϵ_1 and coefficient functions $c_{1,k}$. Then adding the set of vectors $\Omega_1^- = \cup_i(-k_i)$, we have Ω_1 such that (1)-(4) are satisfied. To show that (5) is satisfied, we note that given M of such a form, then

$$M^{12} - M^{21} = M_1^{12} - M_1^{21},$$

and that

$$m_{k_i}^{12} - m_{k_i}^{21} = \frac{25}{169}$$

for each $k_i \in \Omega_1$. Therefore,

$$\begin{aligned}
\sum_{k\in\Omega_1} (c_{1,k}(M))^2 &= \frac{169}{25} \sum_{k\in\Omega_1} (c_{1,k}(M))^2 \frac{25}{169} \\
&= \frac{169}{25} \sum_{k\in\Omega_1} (c_{1,k}(M))^2 \left(m_k^{12} - m_k^{21}\right) \\
&= \frac{169}{25} \sum_{k\in\Omega_1} (c_{1,k}(M_1))^2 \left(m_k^{12} - m_k^{21}\right) \\
&= 2 \sum_{k\in\Omega_1^+} (c_{1,k}(M_1))^2 \\
&= 2 \cdot 5 \cdot \frac{1}{10} \\
&= 1,
\end{aligned}$$

and thus (1)-(5) are satisfied for Ω_1. To construct Ω_2, replace each vector $k = (k_1, k_2, k_3)$ with $k' = (-k_2, k_1, k_3)$. Repeating the previous steps and taking the minimum of ϵ_1 and ϵ_2 finishes the proof. $\square$

With the choice of frequency modes in hand, we can build the following approximately stationary solutions.

Lemma 5.3.5 (Stationary Solutions). *For a finite family of vectors $\Omega \in \mathbb{S}^2$ where $\Omega = -\Omega$, $\lambda \in \mathbb{N}$ such that $\lambda\Omega \in \mathbb{Z}^3$, and constants $c_k \in \mathbb{C}$ indexed by $k \in \Omega$ such that $c_k = \overline{c_{-k}}$, define*

$$\mathbb{V}(x) := \sum_{k\in\Omega} \frac{1}{\lambda} c_k e^{i\lambda k\cdot x}.$$

Then $\mathbb{V}$ is real-valued and there exists Q such that $\overline{\nabla} \cdot \left(\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V}\right) = \operatorname{curl}(Q)$, with Q obeying the bounds

$$\|Q\|_{C^0} \lesssim \left\|(\nabla\mathbb{V})^2\right\|_{C^0}, \qquad \|\operatorname{curl}(Q)\|_{C^0} \lesssim \lambda \left\|(\nabla\mathbb{V})^2\right\|_{C^0}.$$

The mean of $\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V}$ is given by

$$\frac{1}{2}\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V} = \sum_{k\in\Omega} |c_k|^2 \left(k \otimes \overline{k}^{\perp}\right).$$

Furthermore, if $L(z)$ is a smooth function depending only on z, then

$$\begin{aligned}\overline{\nabla} \cdot \left(\nabla(L\mathbb{V}) \otimes \overline{\nabla}^{\perp}(L\mathbb{V})\right) &= L^2\overline{\nabla} \cdot \left(\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V}\right) \\ &= \operatorname{curl}\left(L^2 Q\right) - \left(Q^2\partial_z(L^2), -Q^1\partial_z(L^2), 0\right)^t.\end{aligned}$$

Proof. First note that $\mathbb{V}$ is real-valued by the assumptions on c_k. Then, we have that

$$\Delta(c_k e^{i\lambda k\cdot x}) = -\lambda^2 c_k e^{i\lambda k\cdot x};$$

that is, $\mathbb{V}$ is an eigenfunction of Δ with eigenvalue $-\lambda^2$. In order to show that

$$\overline{\nabla} \cdot \left(\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V}\right)$$

is the curl of a vector field, it suffices to show that the divergence is zero. Calculating the divergence, we have

$$\nabla \cdot \left(\overline{\nabla} \cdot \left(\nabla\mathbb{V} \otimes \overline{\nabla}^{\perp}\mathbb{V}\right)\right) = \overline{\nabla}\nabla\mathbb{V} : \overline{\nabla}^{\perp}\nabla\mathbb{V} + \overline{\nabla}^{\perp}\mathbb{V} \cdot \overline{\nabla}(\Delta\mathbb{V}) = -\lambda^2\overline{\nabla}^{\perp}\mathbb{V} \cdot \overline{\nabla}\mathbb{V} = 0.$$

After writing out $\nabla\mathbb{V}$ and $\overline{\nabla}^{\perp}\mathbb{V}$ in terms of Fourier series with modes k and k', respectively, we can restate this fact in the form of the following algebraic identity which will be crucial later in the paper.

$$\sum_{k,k'\in\Omega} c_k c_{k'} e^{i\lambda(k+k')\cdot x}(i\overline{k}'^{\perp}\cdot ik)(ik\cdot i(k+k'))\lambda^2 = 0 \qquad \forall x\in\mathbb{T}^3. \tag{5.4}$$

The bounds on Q come from noticing that Q solves the elliptic equation

$$Q = (-\Delta)^{-1}\operatorname{curl}\left(\overline{\nabla}\cdot\left(\nabla\mathbb{V}\otimes\overline{\nabla}^{\perp}\mathbb{V}\right)\right)$$

and using the frequency support of $\mathbb{V}$ in conjunction with Lemma D.0.2 to conclude that the singular integral operator $(-\Delta)^{-1}\circ\operatorname{curl}\circ\overline{\nabla}\cdot$ is bounded on C^0. By direct calculation, the low frequency portion of $\nabla\mathbb{V}\otimes\overline{\nabla}^{\perp}\mathbb{V}$ is given as stated.

Given a smooth function $L(z)$, it is clear that

$$\overline{\nabla}\cdot\left(L\overline{\nabla}V)\otimes\overline{\nabla}^{\perp}(L\mathbb{V})\right) = L^2\overline{\nabla}\cdot\left(\overline{\nabla}\mathbb{V}\otimes\overline{\nabla}^{\perp}\mathbb{V}\right)$$

since L depends only on z. We calculate the third component by writing

$$\begin{aligned}\overline{\nabla}\cdot\left(\overline{\nabla}^{\perp}(L\mathbb{V})\partial_z(L\mathbb{V})\right) &= \overline{\nabla}\cdot\left(\overline{\nabla}^{\perp}(L\mathbb{V})\partial_z L\mathbb{V} + \overline{\nabla}^{\perp}(L\mathbb{V})L\partial_z\mathbb{V}\right)\\ &= L\partial_z L\left(\overline{\nabla}^{\perp}\mathbb{V}\cdot\overline{\nabla}\mathbb{V}\right) + L^2\overline{\nabla}^{\perp}\mathbb{V}\cdot\overline{\nabla}(\partial_z\mathbb{V})\\ &= L^2\overline{\nabla}\cdot\left(\partial_z\mathbb{V}\overline{\nabla}^{\perp}\mathbb{V}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\nabla}\cdot\left(\nabla(L\mathbb{V})\otimes\overline{\nabla}^{\perp}(L\mathbb{V})\right) &= L^2\overline{\nabla}\cdot\left(\nabla\mathbb{V}\otimes\overline{\nabla}^{\perp}\mathbb{V}\right)\\ &= L^2\operatorname{curl}(Q)\\ &= \operatorname{curl}(L^2Q) - \left(Q^2\partial_z(L^2), -Q^1\partial_z(L^2), 0\right)^t\end{aligned}$$

after recalling that L depends on z only. □

5.4 Convex Integration Scheme

5.4.1 Inductive Assumptions

We assume the existence of a triple $(\nabla\Psi_q, Q_q, \mathring{M}_q)$ solving

$$\partial_t(\nabla\Psi_q) + \overline{\nabla}\cdot\left(\nabla\Psi_q\otimes\overline{\nabla}^{\perp}\Psi_q\right) = \operatorname{curl}(Q_q) + \overline{\nabla}\cdot\mathring{M}_q. \tag{5.5}$$

The gradient of the stream function $\nabla\Psi_q$, the curl Q_q, and the matrix field $\mathring{M}_q$ are assumed to be supported in frequency in the set

$$\{(x,y) : |(x,y)| \leq \lambda_q\} \times \mathbb{Z}. \tag{5.6}$$

The gradient of the stream function $\nabla\Psi_q$, the curl Q_q, and the matrix field $\mathring{M}_q$ are assumed to be supported in space in the set

$$\mathbb{T}^2 \times [\frac{1}{l_q}, 2\pi - \frac{1}{l_q}]. \tag{5.7}$$

We assume that

$$\|\nabla\Psi_q\|_{C^0} \lesssim 1, \qquad \|\nabla\Psi_q\|_{C^n} \leq \delta_q^{\frac{1}{2}}\lambda_q^n \quad \forall n \geq 1. \tag{5.8}$$

We assume that $\mathring{M}_q$ satisfies

$$\left\|\mathring{M}_q\right\|_{C^0} \leq \eta\delta_{q+1}, \qquad \left\|\mathring{M}_q\right\|_{C^1} \leq \delta_{q+1}\lambda_q. \tag{5.9}$$

In addition, we assume that the material derivative of $\mathring{M}_q$ satisfies

$$\left\|\left(\partial_t + \overline{\nabla}^\perp\Psi_q \cdot \overline{\nabla}\right)\mathring{M}_q\right\|_{C^0} \leq \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q. \tag{5.10}$$

We assume that Q_q satisfies

$$\|Q\|_{C^0} \lesssim 1, \qquad \|\nabla Q\|_{C^0} \leq \delta_q\lambda_q. \tag{5.11}$$

Concerning the prescribed energy profile $e(t)$, we assume that

$$0 \leq e(t) - \int_{\mathbb{T}^3} |\nabla\Psi(t)|^2\, dx \leq \delta_{q+1} \tag{5.12}$$

and

$$e(t) - \int_{\mathbb{T}^3} |\nabla\Psi(t)|^2\, dx \leq \frac{\delta_{q+1}}{8} \Rightarrow \mathring{M}_q(\cdot, t) \equiv 0. \tag{5.13}$$

The bulk of the paper consists of verifying that we can construct a triple $(\nabla\Psi_{q+1}, Q_{q+1}, \mathring{M}_{q+1})$ satisfying (5.5)-(5.13) with q replaced with $q+1$ and parameters $\delta_{q+1} < \delta_q$, $\lambda_{q+1} > \lambda_q$, and l_q, where $\delta_q \to 0$ and $\lambda_q \to 0$ as $q \to \infty$ at rates implying the desired level of Hölder regularity and $l_q \to \infty$ in a way such that everything vanishes at $z = 0$ and $z = 2\pi$.

5.4.2 Velocity Perturbation

5.4.2.1 A Spatial Localizer, Time Partition, Transport

Define the cutoff function L_{q+1} to be a smooth function depending only on z which satisfies

$$0 \le L_{q+1}(z) \le 1, \quad L_{q+1} = 1 \quad \forall (x,y,z) \in \mathbb{T}^2 \times [\frac{1}{l_{q+1}}, 2\pi - \frac{1}{l_{q+1}}], \tag{5.14}$$
$$\|\partial_z L_{q+1}\|_{C^0} \lesssim l_{q+1}, \quad \operatorname{supp} L_{q+1} \subset \mathbb{T}^2 \times [\frac{1}{l_{q+2}}, 2\pi - \frac{1}{l_{q+2}}].$$

Let $\mathcal{X} \in C_c^\infty\left((-\frac{3}{4}, \frac{3}{4})\right)$ be a smooth positive cutoff function such that

$$\sum_{l\in\mathbb{Z}} \mathcal{X}^2(x-l) = 1$$

for all $t \in \mathbb{R}$. Let the support of the energy profile $e(t)$ be contained in a ball of radius R. For μ_{q+1} a large parameter to be specified later and $l \in \mathbb{Z} \cap [-R\mu_{q+1}, R\mu_{q+1}]$, define (we neglect to indicate the dependence on q for ease of notation)

$$\mathcal{X}_l(t) := \mathcal{X}(\mu_{q+1}t - l).$$

Define

$$\rho(t) := \left(\int_{\mathbb{T}^3} L_{q+1}^2\right)^{-1} \min\left(e(t) - \int_{\mathbb{T}^3} \|\nabla\Psi_q\|^2 - \frac{\delta_{q+2}}{2}, 0\right), \qquad \rho_l = \rho(\frac{l}{\mu_{q+1}}). \tag{5.15}$$

By the assumptions (5.12) and (5.13), we have that

$$\rho_l \le \delta_{q+1}, \qquad \rho_l \ne 0 \Rightarrow \rho_l \ge \frac{\delta_{q+1}}{16}. \tag{5.16}$$

Let $\phi_q(z)$ be a mollifier in z which is compactly supported in a ball of radius $\ell^{-1} = \lambda_q^{-\frac{3}{4}}\lambda_{q+1}^{-\frac{1}{4}}$. Define

$$\mathring{M}_{q,\ell} = \mathring{M}_q * \phi_q$$

so that the spatial support of $\mathring{M}_{q,\ell}$ is still contained in the region where $L_{q+1} \equiv 1$ and

$$\left\|\mathring{M}_{q,\ell}\right\|_{C^0} \le \eta\delta_{q+1}, \qquad \left\|\mathring{M}_{q,\ell}\right\|_{C^1} \le \delta_{q+1}\lambda_q, \qquad \left\|\mathring{M}_{q,\ell}\right\|_{C^n} \le \delta_{q+1}\lambda_q\ell^{n-1} \quad \forall n \ge 2.$$

Let $\mathring{M}_{q,l}$ be the unique solution to the transport equation

$$\begin{cases} \partial_t \mathring{M}_{q,l} + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\mathring{M}_{q,l} = 0 \\ \mathring{M}_{q,l}(x, \frac{l}{\mu_{q+1}}) = \mathring{M}_{q,\ell}(x, \frac{l}{\mu_{q+1}}), \end{cases}$$

and set

$$M_{q,l} := \rho_l M_j - \mathring{M}_{q,l}$$

where M_j comes from Lemma 5.3.4, and j is chosen so that the parity of l and j matches. Next, let $\Phi_l : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the solution of

$$\begin{cases} \partial_t \Phi_l + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\Phi_l = 0 \\ \Phi_l(x, \frac{l}{\mu}) = x \end{cases}$$

so that $\Phi_l(\cdot, t)$ is a diffeomorphism of $\mathbb{T}^3$ onto itself, and for $(t, x) \in \mathbb{R} \times \mathbb{T}^3$ the map

$$(x, t) \to e^{i\lambda_{q+1} k \cdot \Phi_l(x,t)}$$

is well-defined.

5.4.2.2 The Perturbation

Apply Lemma 5.3.4 to obtain sets of frequency modes Ω_1 and Ω_2. Let $k \in \Omega = \Omega_1 \cup \Omega_2$ denote a chosen frequency mode. Now define

$$\mathcal{X}_l(t) := \mathcal{X}\left(\mu_{q+1}(t - l)\right)$$

$$a_{kl}(x,t) := \begin{cases} \sqrt{\rho_l} c_{j,k}\left(\frac{M_{q,l}(x,t)}{\rho_l}\right) & \text{if } \rho_l \neq 0 \\ 0 & \text{if } \rho_l = 0 \end{cases}$$

$$w_{kl}(x,t) := a_{kl}(x,t) e^{i\lambda_{q+1} k \cdot \Phi_l(x,t)} ik.$$

where $j = 1$ and $k \in \Omega_1$ if l is odd, and $j = 2$ and $k \in \Omega_2$ if l is even. We must check that a_{kl} is well-defined when $\rho_l \neq 0$. It suffices to check that given ϵ as in Lemma 5.3.4,

$$\frac{\mathring{M}_{q,l}}{\rho_l} < \epsilon.$$

By (5.16), we have that if ρ_l is non-zero, then ρ_l is bounded below by

$$\frac{\delta_{q+1}}{16}.$$

Then

$$\frac{\mathring{M}_{q,l}}{\rho_l} \leq \frac{\eta\delta_{q+1}}{\frac{\delta_{q+1}}{16}}$$

which is less than ϵ as long as η is small enough. $\nabla\mathbb{W}_{q+1}$ is now well-defined by (using the definition of $\mathbb{P}^{\nabla}_{q+1,k}$ given in Definition D.0.2 of Appendix D)

$$\nabla\mathbb{W}_{q+1}(x,t) := \sum_{l \text{ odd}, k\in\Omega_1} \mathbb{P}^{\nabla}_{q+1,k}\big(\mathcal{X}_l(t) w_{kl}(x,t)\big) \quad + \sum_{l \text{ even}, k\in\Omega_2} \mathbb{P}^{\nabla}_{q+1,k}\big(\mathcal{X}_l(t) w_{kl}(x,t)\big).$$

Throughout the rest of the paper, we will simply write

$$\nabla\mathbb{W}_{q+1}(x,t) = \sum_{l,k} \mathbb{P}^{\nabla}_{q+1,k}\big(\mathcal{X}_l(t) w_{kl}(x,t)\big)$$

for the sake of simplicity. The perturbation is then defined by $\nabla(\mathbb{W}_{q+1}L_{q+1})$.

5.4.3 Adding the Perturbation

Define $\nabla\Psi_{q+1} = \nabla\Psi_q + \nabla(\mathbb{W}_{q+1}L_{q+1})$. Using that $\nabla\Psi_q$ solves

$$\partial_t(\nabla\Psi_q) + \overline{\nabla}\cdot\left(\nabla\Psi_q \otimes \overline{\nabla}^{\perp}\Psi_q\right) = \operatorname{curl}(Q_q) + \overline{\nabla}\cdot\mathring{M}_q,$$

we have that $\nabla\Psi_{q+1}$ solves

$$\begin{aligned}
\partial_t(\nabla\Psi_{q+1}) + \overline{\nabla}\cdot\left(\nabla\Psi_{q+1} \otimes \overline{\nabla}^{\perp}\Psi_{q+1}\right) &= \operatorname{curl}(Q_q) \\
&+ \partial_t(\nabla(\mathbb{W}_{q+1}L_{q+1})) + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\nabla(\mathbb{W}_{q+1}L_{q+1}) && \text{(Transport Error)} \\
&+ \overline{\nabla}^{\perp}(\mathbb{W}_{q+1}L_{q+1}) \cdot \overline{\nabla}\nabla\Psi_q && \text{(Nash Error)} \\
&+ \overline{\nabla}\cdot\left(\nabla(\mathbb{W}_{q+1}L_{q+1}) \otimes \overline{\nabla}^{\perp}(\mathbb{W}_{q+1}L_{q+1})\right) + \overline{\nabla}\cdot\mathring{M}_q && \text{(Oscillation Error)} \\
&= \operatorname{curl}(Q_{q+1}) + \overline{\nabla}\cdot\left(\mathring{M}_{q+1}\right).
\end{aligned}$$

The definition of the matrix field $\mathring{M}_{q+1}$ and the vector field Q_{q+1} will be specified in the following sections.

5.4.4 Choice of Parameters

We define the parameters λ_q, δ_q, μ_{q+1}, and l_q for all $q \in \mathbb{N}$ in terms of a real number $c > \frac{5}{2}$, a real number $b > 1$, and a large integer $a \in 13\mathbb{Z}$. The numbers c, b, and a are chosen in that order after first fixing a desired Hölder regularity level $\zeta \in (0, \frac{1}{5})$.

$$\lambda_q := a^{cb^q}, \qquad \delta_q := a^{-b^q}, \qquad \mu_{q+1} := \delta_q^{\frac{1}{4}}\delta_{q+1}^{\frac{1}{4}}\lambda_q^{\frac{1}{2}}\lambda_{q+1}^{\frac{1}{2}}, \qquad l_q := \frac{1}{2^{q+1}}$$

In addition, we implement small parameters

$$0 < \alpha \ll \beta \ll 1$$

which are essentially used to control singular integral operators on L^∞ and to quantify the super-exponential growth of the λ_q's. With these choices, the following inequalities hold.

Lemma 5.4.1 (Parameter Inequalities). *Given λ_q, δ_q, μ_{q+1}, and l_q as defined above, the following inequalities are true for satisfactory choices of c, b, a, β, and α.*

1. $\frac{\delta_q^{\frac{1}{2}}\lambda_q}{\mu_{q+1}} \leq \lambda_{q+1}^{-\beta}$
2. $\mu_{q+1}\delta_{q+1}^{\frac{1}{2}} \leq \delta_{q+2}\lambda_{q+1}$.
3. $l_{q+1}\frac{\delta_{q+1}}{\lambda_{q+1}} \leq \eta\delta_{q+2}$
4. $\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$
5. $\lambda_q^{1+\alpha} \leq \lambda_{q+1}^{1-\alpha}$

Proof. Writing out (1), we see that it is satisfied provided that

$$\frac{a^{-\frac{1}{2}b^q}a^{cb^q}}{a^{-\frac{1}{4}b^q}a^{\frac{1}{2}cb^q}a^{-\frac{1}{4}b^{q+1}}a^{\frac{1}{2}cb^{q+1}}} \leq a^{-\beta cb^{q+1}}.$$

Taking the logarithm in a of both sides and dividing by b^q yields

$$\begin{aligned} -\frac{1}{2} + c + \frac{1}{4} + \frac{1}{4}b - \frac{1}{2}c - \frac{1}{2}cb &\leq -\beta cb \\ \iff b\left(\frac{1}{4} - \left(\frac{1}{2} - \beta\right)c\right) + \frac{1}{2}c - \frac{1}{4} &\leq 0, \end{aligned}$$

which is true if β is small enough. The second inequality is true provided that

$$a^{-\frac{1}{4}b^q}a^{\frac{1}{2}cb^q}a^{-\frac{1}{4}b^{q+1}}a^{\frac{1}{2}cb^{q+1}}a^{-\frac{1}{2}b^{q+1}} \leq a^{-b^{q+2}}a^{cb^{q+1}}.$$

Taking logarithms in a and dividing by b^q again gives

$$-\frac{1}{4}+\frac{1}{2}c-\frac{1}{4}b+\frac{1}{2}cb-\frac{1}{2}b \leq -b^2+cb$$
$$\iff b^2-b\left(\frac{3}{4}+\frac{1}{2}c\right)-\frac{1}{4}+\frac{1}{2}c \leq 0,$$

which is true provided b is close enough to 1. The inequality in (3) follows from the (merely) exponential growth of l_{q+1}. The proof of (4) proceeds similarly to that of (2). (5) follows from the super exponential growth of λ_q provided α is small enough. $\square$

5.4.5 Inductive Step

The proofs of Theorem 5.1.1 and Theorem 5.1.2 will require the following inductive propositions.

Proposition 5.4.2 (**3D QG Inductive Proposition**). *Let $e(t) : \mathbb{R} \to [0,\infty)$ be a smooth, compactly supported energy profile. Then given $c > \frac{5}{2}$, there exists $b > 1$, $a \gg 1$ such that the following holds. Given a triple $\left(\nabla\Psi_q, \mathring{M}_q, Q_q\right)$ satisfying the inductive assumptions* (5.5)-(5.13) *with parameters δ_q, λ_q, l_q defined in terms of a, b, and c, there exists a new triple $\left(\nabla\Psi_{q+1}, \mathring{M}_{q+1}, Q_{q+1}\right)$ satisfying* (5.5)-(5.13) *with q replaced by $q+1$.*

Proposition 5.4.3 (**2D Euler Inductive Proposition**). *With the additional assumption that the matrix field $\mathring{M}_q$ is of the block form*

$$\mathring{M}_q = \begin{bmatrix} m_1 & m_2 & 0 \\ m_2 & -m_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the elimination of any restrictions on the spatial support, the outcome of Proposition 5.4.2 can be achieved while simultaneously prescribing that

$$\partial_z\left(\Psi_{q+1}-\Psi_q\right) \equiv 0.$$

In particular, if $\mathring{M}_1$ is of such a block form, then one can impose that $\partial_z\Psi_q \equiv 0$ for all $q \in \mathbb{N}$.

5.5 Error Estimates

Before estimating the transport, Nash, and oscillation errors, we show the following bounds on the perturbation and $\nabla\Psi_q$.

Lemma 5.5.1 (Preliminary Estimates). *Using the definitions given in the previous section for each function and parameter, the following hold.*

1. $\left\|\nabla^k a_{kl}\right\|_{C^0(\operatorname{supp}\mathfrak{X}_l)} \leq \delta_{q+1}^{\frac{1}{2}}\lambda_q \ell^{k-1}$ *for* $k \in \mathbb{N}$.

2. *For* $t \in \operatorname{supp}\mathfrak{X}_l$, $\|D\Phi_l - \mathrm{Id}\|_{C^0} \leq \frac{\delta_q^{\frac{1}{2}}\lambda_q}{\mu_{q+1}}$ *and* $\left\|\nabla^N \Phi_l\right\|_{C^0} \leq \frac{\delta_q^{\frac{1}{2}}\lambda_q^N}{\mu_{q+1}}$ *when* $N \geq 2$.

3. $\|\nabla e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\|_{C^0(\operatorname{supp}\mathfrak{X}_l)} \leq \lambda_{q+1}^{1-\beta}$ *and* $\left\|\nabla^k e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right\|_{C^0} \leq \lambda_{q+1}^{k(1-\beta)}$ *for* $k \in \mathbb{N}$.

4. $\|D_{t,q}(\nabla\Psi_q)\|_{C^0} \leq \delta_q\lambda_q$.

5. $\|w_{kl}\|_{C^1(\operatorname{supp}\mathfrak{X}_l)} \leq \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$.

6. $\|\nabla(L_{q+1}\mathbb{W}_{q+1})\|_{C^n} \leq \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^n$.

Proof. 1. Using the chain rule estimates in Lemma D.0.5, we can write

$$\begin{aligned}
\|\nabla a_{kl}\|_{C^0(\operatorname{supp}\mathfrak{X}_l)} &= \left\|\nabla\left(\sqrt{\rho_l}c_{j,k}\left(\frac{M_{q,l}}{\rho_l}\right)\right)\right\|_{C^0} \\
&\leq \sqrt{\rho_l}\left(\|\nabla c_{j,k}\|_{C^0}\|\nabla M_{q,l}\|_{C^0}\rho_l^{-1}\right) \\
&\leq C(c_{j,k})\sqrt{\rho_l}\left(\frac{\eta\delta_{q+1}\lambda_q}{\rho_l}\right) \\
&\leq \delta_{q+1}^{\frac{1}{2}}\lambda_q
\end{aligned}$$

We have used here the lower bound $\rho_l \geq \frac{\delta_{q+1}}{8}$, the smoothness of the functions $c_{j,k}$, and a small choice of η. For the second bound, arguing as before and using the C^k bounds on $\nabla\Psi_q$ and $\mathring{M}_{q,\ell}$, and therefore $\mathring{M}_{q,l}$ and $M_{q,l}$, gives the claim.

2. Applying Lemma D.0.5 and the transport estimates in Lemma D.0.4 , we have that

$$\|D\Phi_l - \mathrm{Id}\,\|_{C^0} \leq (t-t_0)\|\overline{\nabla\nabla}^\perp\Psi_q\|_{C^0}e^{(t-t_0)\|\overline{\nabla\nabla}^\perp\Psi_q\|_{C^0}} \leq \frac{\delta_q^{\frac{1}{2}}\lambda_q}{\mu_{q+1}}.$$

The last estimate follows again from Lemma D.0.4 and the C^n bounds of the velocity $\overline{\nabla}^\perp\Psi_q$.

3. We can use (2) and Lemma D.0.5 to calculate

$$\begin{aligned}\left\|\nabla\left(e^{i\lambda_{q+1}(\Phi_l(x,t)-x)\cdot k}\right)\right\|_{C^0(\operatorname{supp}\chi_l)} &\leq \left(\|\nabla e^{ix}\|_{C^0}\left\|\nabla\left(i\lambda_{q+1}k\cdot(\Phi_l-x)\right)\right\|_{C^0}\right)\\ &\leq \lambda_{q+1}\left\|D\Phi_l-\mathrm{Id}\right\|_{C^0}\\ &\leq \frac{\delta_q^{\frac{1}{2}}\lambda_q}{\mu_{q+1}}\\ &\leq \lambda_{q+1}^{-\beta}.\end{aligned}$$

The second claim follows from the C^n bounds on $\overline{\nabla}^{\perp}\Psi_q$, the chain rule Lemma D.0.5, and the transport estimates Lemma D.0.4.

4. We have that $\nabla\Psi_q$ satisfies the transport equation

$$\partial_t(\nabla\Psi_q)+\overline{\nabla}^{\perp}\Psi_q\cdot\overline{\nabla}\nabla\Psi_q=\overline{\nabla}Q_{3,p}+\overline{\nabla}\cdot\mathring{M}_q.$$

By the inductive assumptions (5.9) and (5.11),

$$\|\overline{\nabla}Q_{3,p}\|\leq\delta_q\lambda_q,\qquad \|M\|_{C^1}\leq 4\eta\delta_{q+1}\lambda_q$$

which yields the claim since $\delta_{q+1}\leq\delta_q$.

5. Using that $D_{t,q}w_{kl}=0$ and that $w_{kl}=a_{kl}e^{i\lambda_{q+1}k\cdot x}ik$ at $t=\frac{l}{\mu_{q+1}}$, we apply Lemma D.0.4 to obtain that

$$\|w_{kl}\|_{C^1}\leq\left(\|a_{kl}\|_{C^1}+\|a_{kl}\|_{C^0}\lambda_{q+1}\right)e^{\frac{\|\overline{\nabla}^{\perp}\Psi_q\|_{C^1}}{\mu_{q+1}}}\leq\delta_{q+1}^{\frac{1}{2}}\lambda_q+\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1},$$

proving the result.

6. Applying the Leibniz rule to $\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)$, using the compact frequency support of $\nabla\mathbb{W}_{q+1}$, and noticing that $\nabla^n L_{q+1}=(\partial_z)^n L_{q+1}\ll\lambda_{q+1}^n$ due to the fact that $l_{q+2}\ll\lambda_{q+1}$ gives the claim.

□

5.5.1 Transport Error

Lemma 5.5.2. *The transport error*

$$\partial_t \left(\nabla(\mathbb{W}_{q+1}L_{q+1})\right) + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\nabla(\mathbb{W}_{q+1}L_{q+1})$$

is equal to

$$\operatorname{curl}(Q_T) + \overline{\nabla} \cdot \mathring{M}_T$$

with the estimates

$$\|Q_T\|_{C^0} \leq \delta_{q+1}, \qquad \|Q_T\|_{C^1} \leq \delta_{q+1}\lambda_{q+1}$$

$$\|\mathring{M}_T\|_{C^0} \leq \eta\delta_{q+2}, \qquad \|\mathring{M}_T\|_{C^1} \leq \delta_{q+2}\lambda_{q+1}, \qquad \|D_{t,q}\mathring{M}_T\|_{C^0} \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.$$

Furthermore, Q_T and $\mathring{M}_T$ are supported in the set

$$\mathbb{T}^2 \times \left[\frac{1}{l_{q+1}}, 2\pi - \frac{1}{l_{q+1}}\right].$$

Proof. By the compact support in x and y frequency modes of $\nabla\Psi_q$ and the support in frequency of $\nabla\left(L\mathbb{W}_{q+1}\right)$ in a cylinder whose base is an annulus in x and y centered around λ_{q+1}, the x and y frequency modes of the transport error are supported in the cylinder above an annulus of radius λ_{q+1} in $\mathbb{Z}^2$. Therefore, we can apply the x and y frequency localizer $\bar{\mathbb{P}}_{\approx\lambda_{q+1}}$ and Lemma D.0.3 to write the transport error as

$$\begin{aligned}
&\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\Big(\partial_t(\nabla(\mathbb{W}_{q+1}L_{q+1})) + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\nabla(\mathbb{W}_{q+1}L_{q+1})\Big) \\
&\quad = \bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(L_{q+1}D_{t,q}\nabla\mathbb{W}_{q+1} + (0,0,\partial_z L_{q+1}\partial_t\mathbb{W}_{q+1})^t\right) \\
&\quad := M_{T,1} + M_{T,2}.
\end{aligned}$$

Beginning with $M_{T,1}$, we can commute L_{q+1} and $\bar{\mathbb{P}}_{\approx\lambda_{q+1}}$ and introduce the commutator

$\left[D_{t,q}, \mathbb{P}^{\nabla}_{q+1,k}\right]$ to write

$$
\begin{aligned}
\|M_{T,1}\|_{C^0} &\leq \left\|\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\left[D_{t,q}, \mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl}) + \partial_t \mathcal{X}_l w_{kl}\right)\right\|_{C^0} \\
&\leq \left\|\bar{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k,l} \left[\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}_{q+1,k}\right](w_{kl}\mathcal{X}_l)\right\|_{C^0} + \left\|\bar{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k,l} \mathbb{P}_{q+1,k}\left(\partial_t \mathcal{X}_l w_{kl}\right)\right\|_{C^0} \\
&\lesssim \left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^1} \|w_{kl}\mathcal{X}_l\|_{C^0} + \left\|\partial_t \mathcal{X}_l a_{kl} e^{i\lambda_{q+1}\Phi_l \cdot x}\right\|_{C^0} \\
&\lesssim \delta_q^{\frac{1}{2}} \lambda_q \delta_{q+1}^{\frac{1}{2}} + \mu_{q+1}\delta_{q+1}^{\frac{1}{2}} \\
&\lesssim \mu_{q+1}\delta_{q+1}^{\frac{1}{2}} \\
&\leq \eta \delta_{q+2}\lambda_{q+1}
\end{aligned}
$$

after applying the commutator estimate (D.1). We then decompose $M_{T,1}$ using $\mathbb{P}^{\overline{\nabla}}$ and $\mathbb{P}^{\overline{\nabla}^{\perp}}$ as

$$
M_{T,1} = \mathbb{P}^{\overline{\nabla}}(M_{T,1}) + \mathbb{P}^{\overline{\nabla}^{\perp}}(M_{T,1}).
$$

After applying $\mathcal{D}$, we can absorb the first piece into $\mathring{M}_T$, while the second piece becomes part of Q_T after inverting $\overline{\nabla}^{\perp}$. The desired C^0 bounds on M_T and Q_T follow from the presence of the frequency localizer $\bar{\mathbb{P}}_{\approx\lambda_{q+1}}$, the fact that $\mathcal{D}$ and $\left(\overline{\nabla}^{\perp}\right)^{-1}$ are operators of order -1 in x and y, and an application of Lemma D.0.2. To show the C^1 bounds, we write

$$
\nabla\left(\left[\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl})\right) = \left[\nabla\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl}) + \left[\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}^{\nabla}_{q+1,k}\right](\nabla w_{kl}\mathcal{X}_l).
$$

$$
\begin{aligned}
\|\nabla M_{T,1}\|_{C^0} &\leq \|\nabla L_{q+1} D_{t,q} \nabla \mathbb{W}_{q+1}\|_{C^0} + \|L_{q+1}\nabla\left(D_{t,q}\nabla\mathbb{W}_{q+1}\right)\|_{C^0} \\
&\lesssim \|\partial_z L_{q+1}\|_{C^0} \|D_{t,q}\nabla\mathbb{W}_{q+1}\|_{C^0} + \left\|\left[\nabla\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl})\right\|_{C^0} \\
&\qquad + \left\|\left[\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \mathbb{P}^{\nabla}_{q+1,k}\right](\nabla w_{kl}\mathcal{X}_l)\right\|_{C^0} + \left\|\mathbb{P}^{\nabla}_{q+1,k}\left(\partial_t \mathcal{X}_l \nabla w_{kl}\right)\right\|_{C^0} \\
&\leq l_{q+2}\delta_{q+2}\lambda_{q+1} + \left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^2} \|\mathcal{X}_l w_{kl}\|_{C^0} + \|\overline{\nabla}^{\perp}\Psi_q\|_{C^1}\|\mathcal{X}_l w_{kl}\|_{C^1} + \left\|\mathbb{P}^{\nabla}_{q+1,k}\left(\partial_t \mathcal{X}_l \nabla w_{kl}\right)\right\|_{C^0} \\
&\leq \delta_{q+2}\lambda_{q+1}^2 + \delta_q^{\frac{1}{2}}\lambda_q^2 \delta_{q+1}^{\frac{1}{2}} + \delta_q^{\frac{1}{2}}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1} + \mu_{q+1}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1} \\
&\leq \delta_{q+2}\lambda_{q+1}^2.
\end{aligned}
$$

Then using the fact that differentiating and multiplying by L_{q+1} or ∇L_{q+1} commutes with $\mathcal{D}$ and $\left(\overline{\nabla}^{\perp}\right)^{-1}$ and applying Lemma D.0.2 due to the x and y frequency support allows us to divide by λ_{q+1}, proving the claim. The spatial support of each term is satisfactory using the inductive hypothesis (5.7) and the fact that multiplication by L_{q+1} commutes with convolution operators in x and y.

The entirety of the second term $M_{T,2}$ will be absorbed into $\mathring{M}_T$ by applying $\mathcal{D}$. Since multiplication by L_{q+1} and $\partial_z L_{q+1}$ commutes with $\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}$, we have that

$$\|M_{T,2}\|_{C^0} \lesssim \frac{1}{\lambda_{q+1}} \|\partial_z L_{q+1}\|_{C^0} \|\partial_t \mathbb{W}_{q+1}\|_{C^0}.$$

Since $\partial_t \nabla \mathbb{W}_{q+1} = D_{t,q} \nabla \mathbb{W}_{q+1} - \overline{\nabla}^\perp \Psi_q \cdot \overline{\nabla} \nabla \mathbb{W}_{q+1}$, we have that

$$\|\partial_t \nabla \mathbb{W}_{q+1}\|_{C^0} \lesssim \mu_{q+1} \delta_{q+1}^{\frac{1}{2}} + \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}.$$

Noticing that $\mathbb{W}_{q+1} = (-\Delta)^{-1} \nabla \cdot (\nabla \mathbb{W}_{q+1})$ and using the frequency support of $\mathbb{W}_{q+1}$, we can apply Lemma D.0.2 to obtain

$$\|\partial_t \mathbb{W}_{q+1}\|_{C^0} \lesssim \delta_{q+1}^{\frac{1}{2}}.$$

Plugging in this estimate, we obtain

$$\begin{aligned} \|M_{T,2}\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}} l_{q+1} \delta_{q+1}^{\frac{1}{2}} \\ &\leq \eta \delta_{q+2}. \end{aligned}$$

The C^1 bound follows from estimating

$$\begin{aligned} \|\partial_z L_{q+1} \partial_t \mathbb{W}_{q+1}\|_{C^1} &\leq \|\partial_z L_{q+1}\|_{C^1} \|\partial_t \mathbb{W}_{q+1}\|_{C^0} + \|\partial_z L_{q+1}\|_{C^0} \|\partial_t \mathbb{W}_{q+1}\|_{C^1} \\ &\leq l_{q+1}^2 \delta_{q+1}^{\frac{1}{2}} + l_{q+1} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \end{aligned}$$

applying $\mathcal{D}$, using the frequency support in x and y to divide by a factor of λ_{q+1}, and recalling that $l_{q+1} \leq \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}$.

Before beginning to estimate the material derivative $D_{t,q} \mathring{M}_T$, note that $D_{t,q} L_{q+1} = 0$.

The material derivative of the transport error can then be decomposed as

$$
\begin{aligned}
D_{t,q}&\left(\mathcal{D}\circ\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\partial_t(\nabla(\mathbb{W}_{q+1}L_{q+1}))+\overline{\nabla}^{\perp}\Psi_q\cdot\overline{\nabla}(\nabla(\mathbb{W}_{q+1}L_{q+1}))\right)\right)=\\
&L_{q+1}\left[D_{t,q},\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\right](D_{t,q}(\nabla\mathbb{W}_{q+1}))\\
&\quad+L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(\sum_{kl}\mathbb{P}^{\nabla}_{q+1,k}\left(\partial_t\mathcal{X}_l w_{kl}\right)\right)\right)\\
&\quad+L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(\sum_{kl}\left[D_{t,q},\mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl})\right)\right)\\
&\quad+\partial_z L_{q+1}D_{t,q}\left(\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}(0,0,\partial_t\mathbb{W}_{q+1})^t\right)\\
&:=T_1+T_2+T_3+T_4.
\end{aligned}
$$

Beginning with T_1, we have that by the commutator estimate (D.1) and the estimate on the amplitude given above,

$$
\begin{aligned}
\|T_1\|_{C^0}&\lesssim\frac{1}{\lambda_{q+1}}\|\overline{\nabla}^{\perp}\Psi_q\|_{C^1}\|D_{t,q}(\nabla\mathbb{W}_{q+1})\|_{C^0}\\
&\lesssim\frac{1}{\lambda_{q+1}}\delta_q^{\frac{1}{2}}\lambda_q\mu_{q+1}\delta_{q+1}^{\frac{1}{2}}.
\end{aligned}
$$

Using that $\delta_q^{\frac{1}{2}}\lambda_q\le\mu_{q+1}$ and $\frac{\mu_{q+1}^2}{\lambda_{q+1}}\delta_{q+1}^{\frac{1}{2}}\le\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$, we obtain

$$
\|T_1\|_{C^0}\le\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.
$$

Moving on to T_2, we apply (D.1) and estimate the parameters as in T_1 to obtain

$$
\begin{aligned}
\|T_2\|_{C^0}&=\left\|\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left[D_{t,q}\left(\sum_{kl}\mathbb{P}^{\nabla}_{q+1,k}\left(\partial_t\mathcal{X}_l w_{kl}\right)\right)\right]\right\|_{C^0}\\
&=\left\|\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left[\sum_{kl}\left[D_{t,q},\mathbb{P}^{\nabla}_{q+1,k}\right](\partial_t\mathcal{X}_l w_{kl})+\sum_{kl}\mathbb{P}^{\nabla}_{q+1,k}\left(\partial_t^2\mathcal{X}_l w_{kl}\right)\right]\right\|_{C^0}\\
&\lesssim\frac{1}{\lambda_{q+1}}\left(\|\overline{\nabla}^{\perp}\Psi_q\|_{\bar{C}^1}\|\partial_t\mathcal{X}_l w_{kl}\|_{C^0}+\|\partial_t^2\mathcal{X}_l w_{kl}\|_{C^0}\right)\\
&\lesssim\frac{1}{\lambda_{q+1}}\left(\delta_q^{\frac{1}{2}}\lambda_q\mu_{q+1}\delta_{q+1}^{\frac{1}{2}}+\mu_{q+1}^2\delta_{q+1}^{\frac{1}{2}}\right)\\
&\lesssim\frac{1}{\lambda_{q+1}}\mu_{q+1}^2\delta_{q+1}^{\frac{1}{2}}\\
&\le\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.
\end{aligned}
$$

We now estimate the material derivative of T_3. As everything is localized in x and y frequencies in an annulus of radius λ_{q+1}, we estimate the terms inside parentheses directly and then divide by $\frac{1}{\lambda_{q+1}}$ at the end. We write

$$\begin{aligned} D_{t,q}\left(\sum_{kl}\left[D_{t,q},\mathbb{P}^{\nabla}_{q+1,k}\right](\mathcal{X}_l w_{kl})\right) &= \left[D_{t,q},\sum_{kl}\left[D_{t,q},\mathbb{P}^{\nabla}_{q+1,k}\right]\right](\mathcal{X}_l w_{kl}) + \left[D_{t,q},\mathbb{P}^{\nabla}_{q+1,k}\right]\left(D_{t,q}\left(\mathcal{X}_l w_{kl}\right)\right) \\ &=: T_{3,1}+T_{3,2}. \end{aligned}$$

We can estimate $T_{3,2}$ using the commutator estimate (D.1) as

$$\begin{aligned} T_{3,2} &\leq \|\nabla\Psi_q\|_{C^1}\|\partial_t\mathcal{X}_l w_{kl}\|_{C^0} \\ &\leq \delta_q^{\frac{1}{2}}\lambda_q\mu_{q+1}\delta_{q+1}^{\frac{1}{2}} \\ &\leq \mu_{q+1}^2\delta_{q+1}^{\frac{1}{2}}. \end{aligned}$$

Applying $\mathcal{D}$ and dividing by λ_{q+1} gives the desired estimate. For $T_{3,1}$, we apply the iterated commutator estimate (D.2) to obtain

$$\begin{aligned} T_{3,1} &\leq \frac{1}{\lambda_{q+1}}\|\nabla\Psi_q\|_{C^1}^2\|\mathcal{X}_l w_{kl}\|_{C^1} + \|\mathcal{X}_l w_{kl}\|_{C^0}\left(\lambda_{q+1}\left\|D_{t,q}\overline{\nabla}^{\perp}\Psi_q\right\|_{C^0} + \left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^1}^2\right) \\ &\leq \frac{1}{\lambda_{q+1}}\delta_q\lambda_q^2\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1} + \delta_{q+1}^{\frac{1}{2}}\left(\lambda_{q+1}\delta_q\lambda_q + \delta_q\lambda_q^2\right) \\ &\lesssim \delta_q\delta_{q+1}^{\frac{1}{2}}\lambda_q\lambda_{q+1}. \end{aligned}$$

Applying $\mathcal{D}$ and dividing again by λ_{q+1}, we obtain the desired estimate.

Finally, we write T_4 as

$$T_4 = \partial_z L_{q+1}\left[D_{t,q},\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\right](\partial_t\mathbb{W}_{q+1}) + \partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}(\partial_t\mathbb{W}_{q+1})\right).$$

We can estimate the first term using the commutator estimate (D.1) by

$$l_{q+1}\frac{1}{\lambda_{q+1}}\delta_q^{\frac{1}{2}}\lambda_q\delta_{q+1}^{\frac{1}{2}} \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$$

as desired. For the second term, first note that

$$\partial_t\mathbb{W}_{q+1} = D_{t,q}\mathbb{W}_{q+1} - \overline{\nabla}^{\perp}\Psi_q\cdot\overline{\nabla}\mathbb{W}_{q+1}.$$

Handling the second piece of the second term first, we then have that

$$\begin{aligned}
&\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(\overline{\nabla}^{\perp}\Psi_q\cdot\overline{\nabla}\mathbb{W}_{q+1}\right)\right)\right\|_{C^0}\\
&\qquad\le\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(\overline{\nabla}^{\perp}\Psi_q\right)\cdot\overline{\nabla}\mathbb{W}_{q+1}\right)\right\|_{C^0}\\
&\qquad\qquad+\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\overline{\nabla}^{\perp}\Psi_q\cdot D_{t,q}\left(\overline{\nabla}\mathbb{W}_{q+1}\right)\right)\right\|_{C^0}\\
&\lesssim l_{q+1}\frac{1}{\lambda_{q+1}}\left(\delta_q\lambda_q\delta_{q+1}^{\frac{1}{2}}+\mu_{q+1}\delta_{q+1}^{\frac{1}{2}}\right)\\
&\le\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.
\end{aligned}$$

Before beginning to estimate the first piece of the second term, note that

$$\mathbb{W}_{q+1}=(-\Delta)^{-1}\left(\nabla\cdot\left(\mathbb{P}^{\nabla}_{q+1,k}\mathcal{X}_l w_{kl}\right)\right).$$

Denoting the operator $(-\Delta)^{-1}\circ(\nabla\cdot)\circ\mathbb{P}^{\nabla}_{q+1,k}$ by K, we have that K is an order -1 convolution kernel. Therefore, we can write that

$$\begin{aligned}
&\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(D_{t,q}\left(\mathbb{W}_{q+1}\right)\right)\right)\\
&\qquad=\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left[D_{t,q},K\right]\left(\mathcal{X}_l w_{kl}\right)\right)+\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(K(\partial_t\mathcal{X}_l w_{kl})\right)\right).
\end{aligned}$$

The second term is bounded as follows:

$$\begin{aligned}
\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(K(\partial_t\mathcal{X}_l w_{kl})\right)\right)\right\|_{C^0}&\le\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\left[D_{t,q},K\right]\left(\partial_t\mathcal{X}_l w_{kl}\right)\right)\right\|_{C^0}\\
&\qquad+\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}K(\partial_t^2\mathcal{X}_l w_{kl})\right\|_{C^0}\\
&\lesssim l_{q+1}\frac{1}{\lambda_{q+1}}\frac{1}{\lambda_{q+1}}\delta_q^{\frac{1}{2}}\lambda_q\mu_{q+1}\delta_{q+1}^{\frac{1}{2}}+l_{q+1}\frac{1}{\lambda_{q+1}}\frac{1}{\lambda_{q+1}}\mu_{q+1}^2\delta_{q+1}^{\frac{1}{2}}\\
&\le\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.
\end{aligned}$$

Here we have used the presence of $\bar{\mathbb{P}}_{\approx\lambda_{q+1}}$ and Lemma D.0.3 to see that K gains a factor of $\frac{1}{\lambda_{q+1}}$. Then for the first term, we will use the iterated commutator estimate (D.2) again. We can then write

$$
\begin{aligned}
\left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left[D_{t,q},K\right](\mathcal{X}_l w_{kl})\right)\right\|_{C^0} &\le \left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\left[D_{t,q},\left[D_{t,q},K\right]\right](\mathcal{X}_l w_{kl})\right)\right\|_{C^0} \\
&\quad + \left\|\partial_z L_{q+1}\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\left[D_{t,q},K\right](\partial_t\mathcal{X}_l w_{kl})\right)\right\|_{C^0} \\
&\le \|\partial_z L_{q+1}\|_{C^0}\,\lambda_{q+1}^{-1}\left(\lambda_{q+1}^{-2}\|\nabla\Psi_q\|_{C^1}^2\,\|\mathcal{X}_l w_{kl}\|_{C^1} + \|\mathcal{X}_l w_{kl}\|_{C^0}\left(\|D_{t,q}\nabla\Psi_q\| + \lambda_{q+1}^{-1}\|\nabla\Psi_q\|_{C^1}^2\right)\right) \\
&\quad + \|\partial_z L_{q+1}\|_{C^0}\lambda_{q+1}^{-1}\lambda_{q+1}^{-1}\|\nabla\Psi_q\|_{C^1}\|\partial_t\mathcal{X}_l w_{kl}\|_{C^0} \\
&\le l_{q+1}\frac{1}{\lambda_{q+1}}\left(\lambda_{q+1}^{-2}(\delta_q^{\frac12}\lambda_q)^2\delta_{q+1}^{\frac12}\lambda_{q+1} + \delta_{q+1}^{\frac12}\left(\delta_q\lambda_q + \lambda_{q+1}^{-1}(\delta_q^{\frac12}\lambda_q)^2\right)\right) \\
&\quad + l_{q+1}\frac{1}{\lambda_{q+1}^2}\delta_q^{\frac12}\lambda_q\mu_{q+1}\delta_{q+1}^{\frac12} \\
&\le \delta_{q+2}\delta_{q+1}^{\frac12}\lambda_{q+1},
\end{aligned}
$$

concluding the proof. □

5.5.2 Nash Error

Lemma 5.5.3. *The Nash error*

$$\overline{\nabla}\cdot\left(\overline{\nabla}^{\perp}\left(L_{q+1}\mathbb{W}_{q+1}\right)\otimes\nabla\Psi_q\right)$$

is equal to

$$\operatorname{curl}(Q_N) + \overline{\nabla}\cdot\mathring{M}_N$$

with the estimates

$$\|Q_N\|_{C^0}\le\delta_{q+1},\qquad \|Q_N\|_{C^1}\le\delta_{q+1}\lambda_{q+1}$$

$$\|\mathring{M}_N\|_{C^0}\le\eta\delta_{q+2},\qquad \|\mathring{M}_N\|_{C^1}\le\delta_{q+2}\lambda_{q+1},\qquad \|D_{t,q}\mathring{M}_N\|_{C^0}\le\delta_{q+2}\delta_{q+1}^{\frac12}\lambda_{q+1}.$$

Furthermore, Q_N and $\mathring{M}_N$ are supported in the set

$$\mathbb{T}^2\times\left[\frac{1}{l_{q+1}},2\pi-\frac{1}{l_{q+1}}\right].$$

Proof. Due to the spatial support of $\nabla\Psi_q$ and $\nabla(\mathbb{W}_{q+1}L_{q+1})$, the Nash error is equal to

$$\overline{\nabla}\cdot\left(\nabla\Psi_q\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right)$$

and the claim on the spatial support is immediate since we shall only ever convolve in x and y. We calculate the amplitude by writing

$$\begin{aligned}\left\|\overline{\nabla}\cdot\left(\nabla\Psi_q\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right)\right\|_{C^0} &\leq \left\|\overline{\nabla}(\nabla\Psi_q)\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\\ &\leq \delta_q^{\frac{1}{2}}\lambda_q\delta_{q+1}^{\frac{1}{2}}\\ &\leq \eta\delta_{q+2}\lambda_{q+1}.\end{aligned}$$

Decomposing into $\mathbb{P}^{\overline{\nabla}}$ and $\mathbb{P}^{\overline{\nabla}^{\perp}}$ and using Bernstein's inequality as for the transport error shows the desired C^0 bounds on Q_N and $\mathring{M}_N$. The C^1 bounds follow by applying ∇ to the Nash error and noticing that the x and y frequency support $\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q$ is contained in an annulus of radius λ_{q+1}, allowing us to divide by λ_{q+1} after applying $\mathcal{D}$ and $\left(\overline{\nabla}^{\perp}\right)^{-1}$.

Moving now to the material derivative, we use (D.1) to write that

$$\begin{aligned}\left\|D_{t,q}\left(\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\overline{\nabla}\cdot\left(\nabla\Psi_q\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right)\right)\right\|_{C^0} &\leq \left\|\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}}\left(D_{t,q}\left(\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right)\right)\right\|_{C^0}\\ &\quad+\left\|\left[\mathcal{D}\bar{\mathbb{P}}_{\approx\lambda_{q+1}},D_{t,q}\right]\left(\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right)\right\|_{C^0}\\ &\lesssim \frac{1}{\lambda_{q+1}}\left(\left\|D_{t,q}\left(\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right)\right\|_{C^0}+\left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^1}\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right\|_{C^0}\right)\\ &\leq \frac{1}{\lambda_{q+1}}\left(\left\|D_{t,q}\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\left\|\overline{\nabla}\nabla\Psi_q\right\|_{C^0}+\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\left\|D_{t,q}(\overline{\nabla}\nabla\Psi_q)\right\|_{C^0}\right.\\ &\quad\left.+\left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^1}\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right\|_{C^0}\right)\\ &\leq \frac{1}{\lambda_{q+1}}\left(\left\|D_{t,q}\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\left\|\overline{\nabla}\nabla\Psi_q\right\|_{C^0}+\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\left\|\overline{\nabla}D_{t,q}(\nabla\Psi_q)\right\|_{C^0}\right.\\ &\quad\left.+\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\right\|_{C^0}\left\|\nabla\Psi_q\right\|_{C^1}^2+\left\|\overline{\nabla}^{\perp}\Psi_q\right\|_{C^1}\left\|\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\nabla\Psi_q\right\|_{C^0}\right)\\ &\leq \frac{1}{\lambda_{q+1}}\left(\mu_{q+1}\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q+\delta_{q+1}^{\frac{1}{2}}\delta_q\lambda_q^2+\delta_{q+1}^{\frac{1}{2}}\delta_q\lambda_q^2+\delta_{q+1}^{\frac{1}{2}}\delta_q\lambda_q^2+\delta_q^{\frac{1}{2}}\lambda_q\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_q\right)\\ &\lesssim \frac{1}{\lambda_{q+1}}\mu_{q+1}^2\delta_{q+1}^{\frac{1}{2}}\\ &\leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.\end{aligned}$$

□

5.5.3 Oscillation Error

Before defining and estimating the oscillation error, we address the effect of the localizer L_{q+1}. As discussed earlier, L_{q+1} *factors out* of the oscillation error. The interaction of the perturbation $\nabla(\mathbb{W}_{q+1}L_{q+1})$ with itself is given in the term

$$\overline{\nabla}\cdot\Big(\nabla(L_{q+1}\mathbb{W}_{q+1})\otimes\overline{\nabla}^{\perp}(L_{q+1}\mathbb{W}_{q+1})\Big).$$

Since L_q depends only on z, the first two components are equal to

$$L_{q+1}^2\overline{\nabla}\cdot\Big(\overline{\nabla}\mathbb{W}_{q+1}\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\Big).$$

In the third row, we can write that

$$\begin{aligned}\overline{\nabla}\cdot\Big(\overline{\nabla}^{\perp}\left(L_{q+1}\mathbb{W}_{q+1}\right)\partial_z\left(L_{q+1}\mathbb{W}_{q+1}\right)\Big) &= \overline{\nabla}\cdot\Big(\overline{\nabla}^{\perp}\left(L_{q+1}\mathbb{W}_{q+1}\right)\left(\mathbb{W}_{q+1}\partial_z L_{q+1}+L_{q+1}\partial_z\mathbb{W}_{q+1}\right)\Big)\\ &= L_{q+1}\partial_z L_{q+1}\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\mathbb{W}_{q+1}+L_{q+1}^2\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}(\partial_z\mathbb{W}_{q+1})\\ &= L_{q+1}^2\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\cdot\overline{\nabla}\partial_z\mathbb{W}_{q+1},\end{aligned}$$

showing that

$$\overline{\nabla}\cdot\Big(\nabla(L_{q+1}\mathbb{W}_{q+1})\otimes\overline{\nabla}^{\perp}(L_{q+1}\mathbb{W}_{q+1})\Big)=L_{q+1}^2\overline{\nabla}\cdot\Big(\nabla\mathbb{W}_{q+1}\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}\Big). \tag{5.17}$$

By the inductive assumption (5.7) on the spatial support of $\mathring{M}_q$, we have also that

$$\overline{\nabla}\cdot\mathring{M}_q=L_{q+1}^2\overline{\nabla}\cdot\mathring{M}_q.$$

Therefore

$$\overline{\nabla}\cdot\Big(\nabla(L_{q+1}\mathbb{W}_{q+1})\otimes\overline{\nabla}^{\perp}(L_{q+1}\mathbb{W}_{q+1})\Big)+\overline{\nabla}\cdot\mathring{M}_q=L_{q+1}^2\overline{\nabla}\cdot\Big(\nabla\mathbb{W}_{q+1}\otimes\overline{\nabla}^{\perp}\mathbb{W}_{q+1}+\mathring{M}_q\Big). \tag{5.18}$$

We will decompose the right hand side into several terms. The definition of this decomposition as well as the estimates for each piece comprise the remainder of this section. We first collect some preliminary estimates.

Lemma 5.5.4. *The following estimates hold.*

1. *For $\theta\in[0,1]$, $\|w_{kl}\|_{C^\theta}\lesssim\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{\theta}$.*

2. For $\theta \in [0,2]$, $\left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right) \right\|_{C^\theta} \lesssim \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\theta-\beta}$.

3. $\left\| D_{t,q}\left(\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right)\right) \right\|_{C^0} \lesssim \mu_{q+1}\delta_{q+1}^{\frac{1}{2}}$.

Proof. The proof of (1) follows from interpolating

$$\|w_{kl}\|_{C^0} \le \delta_{q+1}^{\frac{1}{2}}, \qquad \|w_{kl}\|_{C^1} \le \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$$

using Lemma 5.5.1 and Definition D.0.1. To prove (2), recall that by Lemma 5.5.1, each derivative on $a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}$ costs a factor of $\lambda_{q+1}^{1-\beta}$. Then we can apply the commutator estimate (D.3) to obtain

$$\begin{aligned}\left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right) \right\|_{C^k} &\lesssim \frac{1}{\lambda_{q+1}} \sum_{0\le j\le k} \left\|\nabla a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right\|_{C^j} \left\|e^{i\lambda_{q+1}k\cdot x} ik\right\|_{C^{k-j}} \\ &\lesssim \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}^{k-\beta}.\end{aligned}$$

The non-integer bounds then follow from interpolation. To prove (3), observe that

$$\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right) = D_{t,q}\left(\mathbb{P}^{\nabla}_{q+1,k}(w_{kl})\right) - D_{t,q}w_{kl}.$$

and use the estimates in the section on the transport error. $\square$

5.5.3.1 Estimates for the High Frequency Portion

Lemma 5.5.5. *The high frequency portion of the oscillation error*

$$L^2_{q+1}\overline{\nabla}\cdot\left(\sum_{k+k'\neq 0} \mathcal{X}_l\mathcal{X}_{l'}\left(\mathbb{P}^{\nabla}_{q+1,k}(w_{kl})\right)\otimes\left(\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right)\right)$$

is equal to

$$\operatorname{curl}(Q_{high}) + \overline{\nabla}\cdot\mathring{M}_{O,high}$$

with the estimates

$$\|Q_{high}\|_{C^0} \le \delta_{q+1}, \qquad \|Q_{high}\|_{C^1} \le \delta_{q+1}\lambda_{q+1}$$

$$\|\mathring{M}_{O,high}\|_{C^0} \le \eta\delta_{q+2}, \qquad \|\mathring{M}_{O,high}\|_{C^1} \le \delta_{q+2}\lambda_{q+1}, \qquad \|D_{t,q}\mathring{M}_{O,high}\|_{C^0} \le \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.$$

Furthermore, Q_{high} and $\overline{\nabla}\cdot\mathring{M}_{O,high}$ are supported in the set

$$\mathbb{T}^2 \times \left[\frac{1}{l_{q+1}}, 2\pi - \frac{1}{l_{q+1}}\right].$$

Proof. Towards obtaining a decomposition, we can apply the frequency localizer $\overline{\mathbb{P}}_{\approx\lambda_{q+1}}$ since $k \neq k'$ and Lemma D.0.3 to write

$$
\begin{aligned}
&L_{q+1}^2 \overline{\nabla} \cdot \sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \mathbb{P}_{q+1,k}^{\nabla} \left(w_{kl} \right) \otimes \overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp} \left(\overline{w_{k'l'}}^{\perp} \right) \\
&= L_{q+1}^2 \overline{\nabla} \cdot \overline{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \Bigg(\left(\left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] \left(e^{i\lambda_{q+1} k \cdot x} ik \right) + w_{kl} \right) \otimes \\
&\qquad\qquad \left(\left[\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'} \right] \left(e^{i\lambda_{q+1} k' \cdot x} i\overline{k}'^{\perp} \right) + \overline{w_{k'l'}}^{\perp} \right) \Bigg) \\
&= L_{q+1}^2 \overline{\nabla} \cdot \overline{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \Bigg(\left(\left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] \left(e^{i\lambda_{q+1} k \cdot x} ik \right) \right) \otimes \left(\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp} \left(\overline{w_{k'l'}}^{\perp} \right) \right) \Bigg) \\
&\quad + L_{q+1}^2 \overline{\nabla} \cdot \overline{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \Bigg(\left(w_{kl} \right) \otimes \left(\left[\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'} \right] \left(e^{i\lambda_{q+1} k' \cdot x} i\overline{k}'^{\perp} \right) \right) \Bigg) \\
&\quad + L_{q+1}^2 \overline{\nabla} \cdot \overline{\mathbb{P}}_{\approx\lambda_{q+1}} \sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \Bigg(\left(w_{kl} \right) \otimes \left(\overline{w_{k'l'}}^{\perp} \right) \Bigg) \\
&:= L_{q+1}^2 \overline{\nabla} \cdot \left(O_{high,1} + O_{high,2} + O_{high,3} \right)
\end{aligned}
$$

The terms $O_{high,1}$ and $O_{high,2}$ are simpler to analyze, while the analysis of $O_{high,3}$ is more delicate and will be separated into its own lemma.

Calculating the amplitude of $O_{high,1}$ and $O_{high,2}$, we apply Lemma 5.5.4 to see that

$$
\begin{aligned}
\left\| O_{high,1} \right\|_{C^0} + \left\| O_{high,2} \right\|_{C^0} &\lesssim \left\| \left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] \left(e^{i\lambda_{q+1} k \cdot x} ik \right) \right\|_{C^0} \left\| \overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp} \left(\overline{w_{k'l'}}^{\perp} \right) \right\|_{C^0} \\
&\qquad + \left\| w_{kl} \right\|_{C^0} \left\| \left[\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'} \right] \left(e^{i\lambda_{q+1} k' \cdot x} i\overline{k}'^{\perp} \right) \right\|_{C^0} \\
&\lesssim \delta_{q+1} \lambda_{q+1}^{-\beta} \\
&\leq \eta \delta_{q+2}.
\end{aligned}
$$

Then we separate $\overline{\nabla} \cdot O_{high,1}$ and $\overline{\nabla} \cdot O_{high,2}$ using the projection operators $\mathbb{P}^{\overline{\nabla}}$ and $\mathbb{P}^{\overline{\nabla}^{\perp}}$ as

$$
L_{q+1}^2 \overline{\nabla} \cdot \left(O_{high,1} + O_{high,2} \right) = L_{q+1}^2 \left(\mathbb{P}^{\overline{\nabla}} \left(\overline{\nabla} \cdot \left(O_{high,1} + O_{high,2} \right) \right) + \mathbb{P}^{\overline{\nabla}^{\perp}} \left(\overline{\nabla} \cdot \left(O_{high,1} + O_{high,2} \right) \right) \right).
$$

Since applying $\mathbb{P}^{\overline{\nabla}}$ gives a vector field with three components, the first two of which are the horizontal gradient $\overline{\nabla}$ of a scalar function, the first term can be plugged into the inverse divergence $\mathcal{D}$ and absorbed in $\mathring{M}_{O,high}$. Applying $\mathbb{P}^{\overline{\nabla}^{\perp}}$ yields a vector field with no third component whose first two components are the perpendicular gradient $\overline{\nabla}^{\perp}$ of a scalar function, and so we absorb this term into curl(Q_{high}). Since multiplication by L_{q+1} commutes with both operators, the claims on the spatial supports of Q_{high} and $\mathring{M}_{O,high}$ follow. The claims on the C^0 and C^1 norm follow as for the transport and Nash errors after using Lemma 5.5.4, applying $\mathcal{D}$ and $\left(\overline{\nabla}^\perp\right)^{-1}$, and using Bernstein's inequality in x and y to divide by λ_{q+1} due to the presence of the $\overline{\mathbb{P}}_{\approx\lambda_{q+1}}$.

We must now calculate the material derivative of the $\mathring{M}_{O,high}$ portion. Using that multiplication by L_{q+1} commutes with $\overline{\nabla}\cdot$, $\mathcal{D}$, and $D_{t,q}$, we can write that

$$
\begin{aligned}
D_{t,q}&\left(L_{q+1}^2\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}\left(\overline{\nabla}\cdot\left(O_{high,1}\right)\right)\right)\\
&= L_{q+1}^2\left[D_{t,q},\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}\circ(\overline{\nabla}\cdot)\circ\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\right]\Bigg(\sum_{k+k'\neq 0}\mathcal{X}_l\mathcal{X}_{l'}\left(\left[\mathbb{P}^{\nabla}_{q+1,k},a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right]\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\right)\\
&\qquad\qquad\otimes\left(\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right)\Bigg)\\
&\quad+L_{q+1}^2\left(\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}\circ(\overline{\nabla}\cdot)\circ\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\right)D_{t,q}\Bigg(\sum_{k+k'\neq 0}\mathcal{X}_l\mathcal{X}_{l'}\left(\left[\mathbb{P}^{\nabla}_{q+1,k},a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right]\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\right)\\
&\qquad\qquad\otimes\left(\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right)\Bigg)\\
&=: I+II.
\end{aligned}
$$

Since $\left(\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}\circ(\overline{\nabla}\cdot)\circ\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\right)$ is an order zero operator in x and y satisfying the kernel assumptions of the commutator estimate (D.1), we can write

$$
\begin{aligned}
\|I\|_{C^0}&\lesssim\|\nabla\Psi_q\|_{C^1}\left\|\left[\mathbb{P}^{\nabla}_{q+1,k},a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right]\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\right\|_{C^0}\left\|\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(w_{k'l'}\right)\right\|_{C^0}\\
&\lesssim\delta_q^{\frac12}\lambda_q\delta_{q+1}\lambda_{q+1}^{-\beta}\\
&\leq\delta_{q+2}\delta_{q+1}^{\frac12}\lambda_{q+1}.
\end{aligned}
$$

Recalling that $\|D_{t,q}\mathcal{X}_l\|_{C^0}\leq\mu_{q+1}$, using part (4) of Lemma 5.5.4, and noticing that the singular integral operator $\left(\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}\circ(\overline{\nabla}\cdot)\circ\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\right)$ is bounded on L^∞ due to the frequency localizer and Lemma D.0.2, we can estimate II by

$$
\begin{aligned}
\|II\|_{C^0} \lesssim{}& \|D_{t,q}\mathcal{X}_l\|_{C^0} \left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right)\right\|_{C^0} \left\| \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} \left(w_{k'l'}\right)\right\|_{C^0} \\
&+ \left\| D_{t,q}\left(\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right)\right)\right\|_{C^0} \left\| \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} \left(w_{k'l'}\right)\right\|_{C^0} \\
&+ \left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] \left(e^{i\lambda_{q+1}k\cdot x} ik\right)\right\|_{C^0} \left\| D_{t,q}\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} \left(w_{k'l'}\right)\right\|_{C^0} \\
\lesssim{}& \mu_{q+1}\delta_{q+1}\lambda_{q+1}^{-\beta} + \mu_{q+1}\delta_{q+1} \\
\leq{}& \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.
\end{aligned}
$$

The estimate for the material derivative of $O_{high,2}$ is similar, and we omit it. $\square$

We must now show that the conclusions of Lemma 5.5.5 hold for the third piece $O_{high,3}$ of the O_{high} error. Before analyzing the $O_{high,3}$ term, we must carefully compute the divergence and determine which pieces of the resulting expression can be absorbed into the error $M_{O,high}$ and which must be absorbed into $\operatorname{curl}(Q_{high})$. The problematic terms arise when the differential operators fall on $e^{i\lambda_{q+1}k\cdot x}$, since picking up a λ_{q+1} makes the resulting term too large to be canceled out by future perturbations. In the context of the Euler equations, the fact that Beltrami flows are stationary solutions provides an algebraic identity which, when deployed at the right time, shows that the problematic terms can be absorbed into the new pressure. In our setting, the same principle holds, although its manifestation appears more technical for two reasons. First, the vector field Q from Lemma 5.3.5 is defined as the solution to an elliptic equation via a composition of several differential and integral operators which we must account for. Secondly, we must carefully keep track of the spatial localizer L_{q+1} throughout the decomposition and subsequent estimates. The localizer gives us building blocks which are only stationary solutions to leading order, leaving some extra error terms to estimate.

Lemma 5.5.6. *The conclusions of Lemma 5.5.5 hold for $L^2_{q+1}\overline{\nabla}\cdot O_{high,3}$.*

Proof. Calculating the divergence (in x and y, i.e. $\overline{\nabla}\cdot$) and setting

$$
f_{klk'l'} = a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} a_{k'l'}e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'},
$$

we have

$$\begin{aligned}
L_{q+1}^2\overline{\nabla}\cdot O_{high,3} &= L_{q+1}^2\overline{\nabla}\cdot\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\left(\sum_{k+k'\neq 0}\mathcal{X}_l\mathcal{X}_{l'}f_{klk'l'}e^{i\lambda_{q+1}k\cdot x}ik\otimes e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp}\right)\\
&= L_{q+1}^2\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\sum_{k+k'\neq 0}\mathcal{X}_l\mathcal{X}_{l'}\left(\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\otimes\left(e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp}\right)\right)\cdot\overline{\nabla}\left(f_{klk'l'}\right)\\
&\quad + L_{q+1}^2\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\sum_{k+k'\neq 0}\mathcal{X}_l\mathcal{X}_{l'}f_{klk'l'}\overline{\nabla}\cdot\left(e^{i\lambda_{q+1}k\cdot x}ik\otimes e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp}\right)\\
&=: L_{q+1}^2 O_{high,3,1} + L_{q+1}^2 O_{high,3,2}.
\end{aligned}$$

The analysis of $O_{high,3,1}$ is simpler due to the fact that the differential operators have landed on $f_{klk'l'}$. Estimating its amplitude, we have that

$$\begin{aligned}
\left\|L_{q+1}^2 O_{high,3,1}\right\|_{C^0} &\lesssim \|\overline{\nabla}f_{klk'l'}\|_{C^0}\\
&\lesssim \|\overline{\nabla}a_{kl}\|_{C^0}\|a_{kl}\|_{C^0} + \|a_{kl}\|_{C^0}^2\left\|\overline{\nabla}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right\|_{C^0}\\
&\lesssim \delta_{q+1}\lambda_q + \delta_{q+1}\lambda_{q+1}^{1-\beta}\\
&\leq \eta\delta_{q+2}\lambda_{q+1}.
\end{aligned}$$

Recalling that multiplication by L_{q+1}^2 commutes with convolution operators and differentiation in x and y, we then decompose $L_{q+1}^2 O_{high,3,1}$ using the $\mathbb{P}^{\overline{\nabla}}$ and $\mathbb{P}^{\overline{\nabla}^{\perp}}$ operators into

$$L_{q+1}^2 O_{high,3,1} = L_{q+1}^2\mathbb{P}^{\overline{\nabla}}\left(O_{high,3,1}\right) + L_{q+1}^2\mathbb{P}^{\overline{\nabla}^{\perp}}\left(O_{high,3,1}\right).$$

The first term can be plugged into the inverse divergence $\mathcal{D}$ and then absorbed into the error $\mathring{M}_{O,high}$, while the second term has zero third component and can be absorbed into $\operatorname{curl}(Q_{high})$. The desired C^0 and C^1 estimates then follow arguing as before.

We now estimate the material derivative of $\mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \left(L_{q+1}^2 O_{high,3,1} \right)$. We write

$$\begin{aligned}
&D_{t,q} \left(L_{q+1}^2 \mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \left(O_{high,3,1} \right) \right) \\
&\quad = L_{q+1}^2 D_{t,q} \left(\mathcal{D} \mathbb{P}^{\overline{\nabla}} \sum_{k+k' \neq 0} \overline{\mathbb{P}}_{\approx \lambda_{q+1}} \mathcal{X}_l \mathcal{X}_{l'} \left(ik \otimes i\overline{k}'^{\perp} e^{i\lambda_{q+1}(k+k')\cdot x} \right) \overline{\nabla} f_{klk'l'} \right) \\
&\quad = L_{q+1}^2 \left[D_{t,q}, \mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}} \right] \left(\sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \left(ik \otimes i\overline{k}'^{\perp} e^{i\lambda_{q+1}(k+k')\cdot x} \right) \overline{\nabla} f_{klk'l'} \right) \\
&\qquad + L_{q+1}^2 \mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}} \left(D_{t,q} \left(\sum_{k+k' \neq 0} \mathcal{X}_l \mathcal{X}_{l'} \left(ik \otimes i\overline{k}'^{\perp} e^{i\lambda_{q+1}(k+k')\cdot x} \right) \overline{\nabla} f_{klk'l'} \right) \right) \\
&\quad =: I + II.
\end{aligned}$$

We bound I using (D.1) and the fact that $\mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}}$ is an order -1 convolution operator in x and y localized in frequency at λ_{q+1}, obtaining

$$\begin{aligned}
\|I\|_{C^0} &\lesssim \|\nabla \Psi_q\|_{C^1} \frac{1}{\lambda_{q+1}} \|\overline{\nabla} f_{klk'l'}\|_{C^0} \\
&\lesssim \delta_q^{\frac{1}{2}} \lambda_q \frac{1}{\lambda_{q+1}} \delta_{q+2} \lambda_{q+1} \\
&\leq \delta_{q+2} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}.
\end{aligned}$$

Before bounding II, we write out

$$\begin{aligned}
\mathcal{X}_l \mathcal{X}_{l'} \overline{\nabla} f_{klk'l'} e^{i\lambda_{q+1}(k+k')\cdot x} &= \mathcal{X}_l \mathcal{X}_{l'} \left(a_{kl} \overline{\nabla} a_{k'l'} + a_{k'l'} \overline{\nabla} a_{kl} \right) e^{i\lambda_{q+1}(k+k')\cdot x} \\
&\quad + i\lambda_{q+1} \mathcal{X}_l \mathcal{X}_{l'} a_{kl} a_{k'l'} \left(\left(D\Phi_l - \mathrm{Id} \right) \overline{k} + \left(D\Phi_{l'} - \mathrm{Id} \right) \overline{k}' \right) e^{i\lambda_{q+1}(k+k')\cdot x}.
\end{aligned}$$

Then computing $D_{t,q}$ of this quantity gives

$$\begin{aligned}
D_{t,q} \left(\overline{\nabla} f_{klk'l'} e^{i\lambda_{q+1}(k+k')\cdot x} \mathcal{X}_l \mathcal{X}_{l'} \right) &= \left(\mathcal{X}_l \mathcal{X}_{l'} \right)' \left(a_{kl} \overline{\nabla} a_{k'l'} + a_{k'l'} \overline{\nabla} a_{kl} \right) e^{i\lambda_{q+1}(k+k')\cdot x} \\
&\quad + i\lambda_{q+1} (\mathcal{X}_l \mathcal{X}_{l'})' a_{kl} a_{k'l'} \left(\left(D\Phi_l - \mathrm{Id} \right) \overline{k} + \left(D\Phi_{l'} - \mathrm{Id} \right) \overline{k}' \right) e^{i\lambda_{q+1}(k+k')\cdot x} \\
&\quad - \mathcal{X}_l \mathcal{X}_{l'} \left(a_{kl} \overline{\nabla} \overline{\nabla}^{\perp} \Psi_q : \overline{\nabla} a_{k'l'} + a_{k'l'} \overline{\nabla} \overline{\nabla}^{\perp} \Psi_q : \overline{\nabla} a_{kl} \right) e^{i\lambda_{q+1}(k+k')\cdot x} \\
&\quad - i\lambda_{q+1} \mathcal{X}_l \mathcal{X}_{l'} a_{kl} a_{k'l'} \left(\overline{\nabla} \overline{\nabla}^{\perp} \Psi_q : D\Phi_l \cdot k + \overline{\nabla} \overline{\nabla}^{\perp} \Psi_q : D\Phi_{l'} \cdot k' \right) e^{i\lambda_{q+1}(k+k')\cdot x}.
\end{aligned}$$

Then we can bound II by

$$\begin{aligned}
\|II\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}} \left(\mu_{q+1} \delta_{q+1} \lambda_{q+1} + \lambda_{q+1} \mu_{q+1} \delta_{q+1} + \delta_q^{\frac{1}{2}} \lambda_q \delta_{q+1} \lambda_q + \lambda_{q+1} \delta_{q+1} \delta_q^{\frac{1}{2}} \lambda_q \right) \\
&\leq \delta_{q+2} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}.
\end{aligned}$$

We now move to the decomposition and estimation of $L^2_{q+1}O_{high,3,2}$. While in general projecting a vector field onto gradients using $\mathbb{P}_\nabla$ induces no gain in regularity, the highest frequency terms in $O_{high,3,2}$ belong to the kernel of the divergence operator. To see this, let us compute the divergence (now in x, y, and z, i.e. $\nabla\cdot$) of $O_{high,3,2}$:

$$\begin{aligned}
&\nabla\cdot\left(\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\sum_{k+k'\neq 0} f_{klk'l'}\overline{\nabla}\cdot\left(e^{i\lambda_{q+1}k\cdot x}ik\otimes e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp}\right)\right)\\
&\quad=\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\sum_{k+k'\neq 0} f_{klk'l'}\left(i\overline{k}'^{\perp}\cdot ik\right)(\lambda_{q+1})^2 e^{i\lambda_{q+1}(k+k')\cdot x}ik\cdot i(k+k')\\
&\qquad+\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\sum_{k+k'\neq 0} i\overline{k}'^{\perp}\cdot ik\left(\lambda_{q+1}e^{i\lambda_{q+1}(k+k')\cdot x}\right)\nabla\left(f_{klk'l'}\right)\cdot ik\\
&\quad=: I+II.
\end{aligned}$$

Since the sum is over $k\in\Omega^1$, $k'\in\Omega^2$ where the parity of l' and l matches that of the corresponding sets Ω^i to which k and k' belong, the coefficients $f_{klk'l'}$ allow for the application of the algebraic identity (5.4) from Lemma 5.3.5. Therefore, I is equal to zero pointwise in $\mathbb{T}^3$, showing that the problematic terms are annihilated by the divergence. Then we can write that

$$\begin{aligned}
\nabla\mathcal{F} &:= \mathbb{P}_\nabla\left(O_{high,3,2}\right)\\
&= \nabla\circ(-\Delta)^{-1}\circ(\nabla\cdot)\left(O_{high,3,2}\right)\\
&= \nabla\circ(-\Delta)^{-1}(II).
\end{aligned}$$

Although the third component of the frequency support of $\mathcal{F}$ is not compact, the first two components are supported in an annulus centered around λ_{q+1}, and so Bernstein's inequality gives that

$$\begin{aligned}
\|\nabla\mathcal{F}\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}}\|II\|_{C^0}\\
&\lesssim \frac{1}{\lambda_{q+1}}\lambda_{q+1}\|f_{klk'l'}\|_{C^1}\\
&\leq \eta\delta_{q+2}\lambda_{q+1}.
\end{aligned}$$

Conversely, after setting $\mathcal{G} := \mathbb{P}_{\mathrm{curl}}(O_{high,3,2})$, we have that

$$\begin{aligned}\|\mathrm{curl}(\mathcal{G})\|_{C^0} &= \|\mathbb{P}_{\mathrm{curl}}(O_{high,3,2})\|_{C^0}\\ &\leq \|O_{high,3,2}\|_{C^0} + \|\nabla\mathcal{F}\|_{C^0}\\ &\lesssim \lambda_{q+1}\|f_{klk'l'}\|_{C^0} + \|\nabla\mathcal{F}\|_{C^0}\\ &\lesssim \delta_{q+1}\lambda_{q+1}.\end{aligned}$$

Furthermore, since $\mathcal{G} = (-\Delta)^{-1}\circ \mathrm{curl}(O_{high,3,2})$ is given by an operator of order -1 applied to $O_{high,3,2}$, by the presence of $\overline{\mathbb{P}}_{\approx\lambda_{q+1}}$ and Bernstein's inequality we see that $\|\mathcal{G}\|_{C^0} \lesssim \delta_{q+1}$.

We are now ready to decompose $L^2_{q+1}O_{high,3,2}$.

$$\begin{aligned}L^2_{q+1}O_{high,3,2} &= L^2_{q+1}\mathbb{P}_{\nabla}\left(O_{high,3,2}\right) + L^2_{q+1}\mathbb{P}_{\mathrm{curl}}(O_{high,3,2})\\ &= L^2_{q+1}\nabla\mathcal{F} + L^2_{q+1}\,\mathrm{curl}(\mathcal{G})\\ &= \begin{bmatrix}\partial_x\left(L^2_{q+1}\mathcal{F}\right)\\ \partial_y\left(L^2_{q+1}\mathcal{F}\right)\\ L^2_{q+1}\partial_z\mathcal{F}\end{bmatrix} + \begin{bmatrix}-\mathcal{G}^2\partial_z(L^2_{q+1})\\ \mathcal{G}^1\partial_z(L^2_{q+1})\\ 0\end{bmatrix} + \mathrm{curl}\left(L^2_{q+1}\mathcal{G}\right).\end{aligned}$$

The first term can now be absorbed into the error $M_{O,high}$ after applying $\mathcal{D}$, while the third term can be absorbed into $\mathrm{curl}(Q_{high})$. The estimates on the amplitudes, C^1 norms, and spatial supports follow from the above estimates on $\mathcal{F}$ and $\mathcal{G}$. Before addressing the second term, which we shall call $\mathcal{L}$, let us calculate the material derivative of the first.

$$\begin{aligned}D_{t,q}\left(\mathcal{D}\left(L^2_{q+1}\nabla\mathcal{F}\right)\right) &= L^2_{q+1}D_{t,q}\left(\mathcal{D}(\nabla\mathcal{F})\right)\\ &= L^2_{q+1}\left[D_{t,q},\mathcal{D}\right]\left(\nabla\mathcal{F}\right) + L^2_{q+1}\mathcal{D}\left(D_{t,q}(\nabla\mathcal{F})\right).\end{aligned}$$

We can bound the first term using (D.1) and the fact that $\nabla\mathcal{F}$ is supported in an annulus of radius λ_{q+1} in x and y frequencies by

$$\begin{aligned}\left\|L^2_{q+1}\left[D_{t,q},\mathcal{D}\right]\left(\nabla\mathcal{F}\right)\right\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}}\|\nabla\Psi_q\|_{C^1}\|\nabla\mathcal{F}\|_{C^0}\\ &\lesssim \delta_q^{\frac{1}{2}}\lambda_q\frac{1}{\lambda_{q+1}}\delta_{q+2}\lambda_{q+1}\\ &\leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.\end{aligned}$$

We decompose the second term further as

$$L^2_{q+1}\mathcal{D}\left(D_{t,q}(\nabla\mathcal{F})\right) = L^2_{q+1}\mathcal{D}\left(\left[D_{t,q},\mathbb{P}_{\nabla}\right]\left(O_{high,3,2}\right)\right) + L^2_{q+1}\mathcal{D}\left(\mathbb{P}_{\nabla}\left(D_{t,q}(O_{high,3,2})\right)\right).$$

Using the fact that $O_{high,3,2}$ is supported in an annulus of size λ_{q+1} in x and y frequencies, we can bound the first term using the commutator estimate from Proposition D.0.6 by

$$\begin{aligned}\left\|L_{q+1}^2\mathcal{D}\left(\left[D_{t,q},\mathbb{P}_\nabla\right]\left(O_{high,3,2}\right)\right)\right\|_{C^0} &\lesssim \frac{1}{\lambda_{q+1}}\|\nabla\Psi_q\|_{C^{1+\alpha}}\|O_{high,3,2}\|_{C^\alpha}\\ &\lesssim \frac{1}{\lambda_{q+1}}\delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\delta_{q+1}\lambda_{q+1}^{1+\alpha}\\ &\leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}\end{aligned}$$

if α is small enough. Then for the second term, we can write

$$\begin{aligned}&\left\|L_{q+1}^2\mathcal{D}\left(\mathbb{P}_\nabla\left(D_{t,q}(O_{high,3,2})\right)\right)\right\|_{C^0}\\ &\quad= \left\|L_{q+1}^2\mathcal{D}\circ\mathbb{P}_\nabla\left(\left[D_{t,q},\overline{\mathbb{P}}_{\approx\lambda_{q+1}}\right]\left(\sum_{k+k'\neq 0}a_{kl}a_{k'l'}e^{i\lambda_{q+1}\Phi_l\cdot k}e^{i\lambda_{q+1}\Phi_l\cdot k'}\lambda_{q+1}k\otimes\overline{k}'^{\perp}(k+k')\right)\right)\right\|_{C^0}\\ &\quad\lesssim \frac{1}{\lambda_{q+1}}\|\nabla\Psi_q\|_{C^1}\lambda_{q+1}\|a_{kl}\|_{C^0}^2\\ &\quad\leq \frac{1}{\lambda_{q+1}}\delta_q^{\frac{1}{2}}\lambda_q\lambda_{q+1}\delta_{q+1}\\ &\quad\leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.\end{aligned}$$

We now return to $\mathcal{L}$. Since $\mathcal{L}$ has derivatives on L_{q+1} rather than $\mathcal{G}$, it is significantly smoother than $\operatorname{curl}(\mathcal{G})$. We decompose $\mathcal{L}$ as

$$\mathcal{L} = \mathbb{P}^{\overline{\nabla}}\left(\mathcal{L}\right) + \mathbb{P}^{\overline{\nabla}^{\perp}}\left(\mathcal{L}\right).$$

Then $\mathbb{P}^{\overline{\nabla}}\left(\mathcal{L}\right)$ is absorbed into the error $M_{O,high}$ after applying $\mathcal{D}$, while $\mathbb{P}^{\overline{\nabla}^{\perp}}\left(\mathcal{L}\right)$ can be absorbed into the curl since it has zero third component. Estimating the amplitude of $\mathcal{L}$, we can write

$$\|\mathcal{L}\|_{C^0} \lesssim \|\partial_z L_{q+1}\|_{C^0}\|\mathcal{G}\|_{C^0} \leq l_{q+1}\delta_{q+1} \leq \eta\delta_{q+2}\lambda_{q+1},$$

and thus the desired C^0 and C^1 estimates follow from Bernstein's inequality and the fact that $\mathcal{L}$ is compactly supported in frequency in x and y.

Finally, it remains to estimate the material derivative of $\mathcal{D}\mathbb{P}^{\overline{\nabla}}\mathcal{L}$.

$$\begin{aligned}
&\left\| \partial_z(L_{q+1}^2) D_{t,q} \left(\mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ (-\Delta)^{-1} \circ \operatorname{curl} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}} \left(\sum_{k+k'\neq 0} a_{kl} a_{k'l'} e^{i\lambda_{q+1}(\Phi_l \cdot k + \Phi_l \cdot k')} k \otimes \overline{k}'^{\perp}(k) \right) \right) \right\|_{C^0} \\
&\leq l_{q+1} \left\| \left[D_{t,q}, \mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ (-\Delta)^{-1} \circ \operatorname{curl} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}} \right] \left(\sum_{k+k'\neq 0} a_{kl} a_{k'l'} e^{i\lambda_{q+1}(\Phi_l \cdot k + \Phi_l \cdot k')} k \otimes \overline{k}'^{\perp}(k) \right) \right\|_{C^0} \\
&\qquad \lesssim l_{q+1} \frac{1}{\lambda_{q+1}^2} \|\nabla \Psi_q\|_{C^{1+\alpha}} \|O_{high,3,2}\|_{C^\alpha} \\
&\qquad \lesssim l_{q+1} \frac{1}{\lambda_{q+1}^2} \delta_q^{\frac{1}{2}} \lambda_q^{1+\alpha} \delta_{q+1} \lambda_{q+1}^{1+\alpha} \\
&\qquad \leq \delta_{q+2} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}.
\end{aligned}$$

We remark that estimating the commutator of $D_{t,q}$ and $\mathcal{D} \circ \mathbb{P}^{\overline{\nabla}} \circ (-\Delta)^{-1} \circ \operatorname{curl} \circ \overline{\mathbb{P}}_{\approx \lambda_{q+1}}$ can be done following the ideas of the proof of (D.1) if one is willing to pay a C^α norm on $\overline{\nabla}^2 \Psi_q$ and $O_{high,3,2}$, which is acceptable considering that l_{q+1} is much smaller than λ_{q+1}. □

5.5.3.2 Estimates for the Low Frequency Portion

O_{low} is given by

$$O_{low} = L_{q+1}^2 \overline{\nabla} \cdot \left(\sum_{k+k'=0} \mathcal{X}_l \mathcal{X}_{l'} \mathbb{P}_{q+1,k}^{\nabla} (w_{kl}) \otimes \overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp} \left(\overline{w_{k'l'}}^{\perp} \right) \right) + L_{q+1}^2 \overline{\nabla} \cdot \mathring{M}_q.$$

Recall that the choice of vectors k implies that if $k = -k'$, then l and l' have the same parity. For l and l' with the same parity, $\sum_{l'} \mathcal{X}_l \mathcal{X}_{l'} = \mathcal{X}_l^2$. In order to isolate the terms which cancel out $\overline{\nabla} \cdot \mathring{M}_q$, we rewrite O_{low} as

$$\begin{aligned}
O_{low} = L_{q+1}^2 \overline{\nabla} \cdot \Bigg(&\sum_{k+k'=0} \mathcal{X}_l^2 \left(\left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] (e^{i\lambda_{q+1} k \cdot x} ik) + w_{kl} \right) \\
&\otimes \left(\left[\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'} \right] (e^{i\lambda_{q+1} k' \cdot x} ik') + \overline{w_{k'l'}}^{\perp} \right) \Bigg) \\
&+ L_{q+1}^2 \overline{\nabla} \cdot \mathring{M}_q \\
=:{}& O_{low,1} + O_{low,2} + O_{low,3} + O_{low,4} + O_{low,5}
\end{aligned}$$

where

$$O_{low,1} := L_{q+1}^2 \overline{\nabla} \cdot \left(\left(\sum_{k+k'=0} \mathcal{X}_l^2 \left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] (e^{i\lambda_{q+1} k \cdot x} ik) \right) \otimes \overline{\mathbb{P}_{q+1,k}^{\nabla}}^{\perp} (\overline{w_{k'l'}}^{\perp}) \right),$$

$$O_{low,2} := L_{q+1}^2 \overline{\nabla} \cdot \left(\sum_{k+k'=0} \mathcal{X}_l^2 w_{kl} \otimes \left(\left[\overline{\mathbb{P}_{q+1,k'}^{\overline{\nabla}}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'}-x)\cdot k'} \right] \left(e^{i\lambda_{q+1}k'\cdot x} i\overline{k}'^{\perp} \right) \right) \right),$$

$$O_{low,3} = L_{q+1}^2 \overline{\nabla} \cdot \left(\mathring{M}_q - \mathring{M}_{q,\ell} \right)$$

$$O_{low,4} := L_{q+1}^2 \overline{\nabla} \cdot \left(\sum_{k+k'=0} \mathcal{X}_l^2 \left(w_{kl} \otimes \overline{w_{k'l'}}^{\perp} - M_{q,l} \right) \right)$$

$$O_{low,5} := L_{q+1}^2 \overline{\nabla} \cdot \left(\sum_{l} \mathcal{X}_l^2 \left(\mathring{M}_{q,\ell} - \mathring{M}_{q,l} \right) \right).$$

We see that by construction,

$$O_{low,4} = L_{q+1}^2 \overline{\nabla} \cdot \left(\frac{1}{2} \sum_{k} \mathcal{X}_l^2 \left(|a_{kl}|^2 k \otimes \overline{k}^{\perp} - M_{q,l} \right) \right) = 0,$$

giving us the required cancellation. Thus, it remains to decompose and estimate $O_{low,1}$, $O_{low,2}$, and $O_{low,3}$, and $O_{low,5}$. We state the results as follows.

Lemma 5.5.7. *The low frequency portion of the oscillation error O_{low} is equal to*

$$\operatorname{curl}(Q_{low}) + \overline{\nabla} \cdot \mathring{M}_{O,low}$$

with the estimates

$$\|Q_{low}\|_{C^0} \leq \delta_{q+1}, \qquad \|Q_{low}\|_{C^1} \leq \delta_{q+1}\lambda_{q+1}$$

$$\|\mathring{M}_{O,low}\|_{C^0} \leq \eta\delta_{q+2}, \qquad \|\mathring{M}_{O,low}\|_{C^1} \leq \delta_{q+2}\lambda_{q+1}, \qquad \|D_{t,q}\mathring{M}_{O,low}\|_{C^0} \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.$$

Furthermore, Q_{low} and $\mathring{M}_{O,low}$ are supported in the set

$$\mathbb{T}^2 \times \left[\frac{1}{l_{q+1}}, 2\pi - \frac{1}{l_{q+1}} \right].$$

Proof. We start by decomposing $O_{low,1}$ as

$$O_{low,1} = \mathbb{P}^{\overline{\nabla}}\left(O_{low,1}\right) + \mathbb{P}^{\overline{\nabla}^{\perp}}\left(O_{low,1}\right).$$

As $\mathbb{P}^{\overline{\nabla}}$ and $\mathbb{P}^{\overline{\nabla}^{\perp}}$ are convolution operators in x and y only, they commute with multiplication by L_{q+1}^2, and the claim on the spatial supports follows. The first term is absorbed into $M_{O,low}$

after applying $\mathcal{D}$, while the second term is absorbed into $\operatorname{curl}(Q_{low})$ by inverting $\overline{\nabla}^{\perp}$. We estimate the $\mathbb{P}^{\overline{\nabla}}$ portion now.

$$
\begin{aligned}
&\|\mathcal{D}\mathbb{P}^{\overline{\nabla}} O_{low,1}\|_{C^0} \\
&\qquad \leq \sup_{k+k'=0} \left\| \mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot \left(\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] (e^{i\lambda_{q+1}k\cdot x} k) \otimes \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} (\overline{w_{k'l'}}^{\perp}) \right) \right\|_{C^0(\operatorname{supp} \mathcal{X}_{l'})} \\
&\qquad \leq \left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] (e^{i\lambda_{q+1}k\cdot x} k) \right\|_{C^0} \left\| \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} \left(\overline{w_{k'l'}}^{\perp}\right) \right\|_{C^\alpha} \\
&\qquad\qquad + \left\| \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] (e^{i\lambda_{q+1}k\cdot x} k) \right\|_{C^\alpha} \left\| \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} \left(\overline{w_{k'l'}}^{\perp}\right) \right\|_{C^0} \\
&\qquad \lesssim \delta_{q+1}\lambda_{q+1}^{\alpha-\beta} \\
&\qquad \leq \eta\delta_{q+2}
\end{aligned}
$$

after using Lemma 5.5.1 and assuming α is sufficiently small. The estimate for the $\mathbb{P}^{\overline{\nabla}^{\perp}}$ portion follows by simply replacing $\mathcal{D}\circ\mathbb{P}^{\overline{\nabla}}$ with $(-\overline{\Delta})^{-1}\circ(\overline{\nabla}^{\perp}\cdot)$ in the above argument.

To calculate the C^1 norms, we write

$$
\begin{aligned}
\left\|\nabla\mathcal{D}\mathbb{P}^{\overline{\nabla}} O_{low,1}\right\|_{C^0} &\lesssim \left\|\partial_z(L_{q+1}^2)\right\|_{C^0} \\
&\qquad \times \sup_{k+k'=0} \left\| \mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot \left(\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] (e^{i\lambda_{q+1}k\cdot x} k) \otimes \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} (\overline{w_{k'l'}}^{\perp}) \right) \right\|_{C^0(\operatorname{supp} \mathcal{X}_{l'})} \\
&\quad + \sup_{k+k'=0} \left\| \mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot \left(\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right] (e^{i\lambda_{q+1}k\cdot x} k) \otimes \overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp} (\overline{w_{k'l'}}^{\perp}) \right) \right\|_{C^1(\operatorname{supp} \mathcal{X}_{l'})} \\
&\lesssim \delta_{q+2} l_{q+1} + \delta_{q+2}\lambda_{q+1} \\
&\leq \eta\delta_{q+2}\lambda_{q+1}
\end{aligned}
$$

after arguing as above. The decomposition and estimate for $O_{low,2}$ is analogous, and we omit the calculation.

Note that $O_{low,3} = L_{q+1}^2 \overline{\nabla}\cdot\left(\mathring{M}_q - \mathring{M}_{q,\ell}\right)$ is already the divergence of a suitable matrix. To estimate the C^0 norm, standard mollification estimates give

$$
\begin{aligned}
\left\|\mathring{M}_q - \mathring{M}_{q,\ell}\right\|_{C^0} &\leq \delta_{q+1}\lambda_q\lambda_q^{-\frac{3}{4}}\lambda_{q+1}^{-\frac{1}{4}} \\
&\leq \eta\delta_{q+2}.
\end{aligned}
$$

The C^1 norm is then easily controlled by $2\left\|\mathring{M}_q\right\|_{C^1} = 2\delta_{q+1}\lambda_q \leq \delta_{q+2}\lambda_{q+1}$, showing the desired result.

For $O_{low,5}$ we recall that for $t = \frac{l}{\mu_{q+1}}$,

$$\mathring{M}_{q,\ell}(t) = \mathring{M}_{q,l}(t)$$

for all $x \in \mathbb{T}^3$, and that

$$D_{t,q}(\mathring{M}_{q,\ell} - \mathring{M}_{q,l}) = D_{t,q}\mathring{M}_{q,\ell}.$$

Before calculating the C^0 and C^1 norm, let us calculate the material derivative of $\mathring{M}_{q,\ell}$.

$$D_{t,q}\mathring{M}_{q,\ell} = \left(D_{t,q}\mathring{M}_q\right) * \phi_q + \overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\mathring{M}_{q,\ell} - \left(\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\mathring{M}_q\right) * \phi_q$$

A simple calculation shows that the commutator

$$\left\|\left[\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}, \phi_q *\right](\mathring{M}_q)\right\|_{C^0} \leq \|\overline{\nabla}^{\perp}\Psi_q\|_{C^1}\|\mathring{M}_q\|_{C^1}\ell^{-1},$$

thus showing that

$$\left\|D_{t,q}\mathring{M}_{q,\ell}\right\|_{C^0} \leq \delta_{q+1}\lambda_q\delta_q^{\frac{1}{2}}\lambda_q\ell^{-1} \leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.$$

In addition, we obtain that

$$\left\|D_{t,q}\mathring{M}_{q,\ell}\right\|_{C^1} \leq \delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\ell.$$

Applying the transport estimate from Lemma D.0.4 and the inductive assumption (5.10), we find

$$\begin{aligned}
\left\|L_{q+1}^2\sum_l \mathcal{X}_l^2\left(\mathring{M}_{q,\ell} - \mathring{M}_{q,l}\right)\right\|_{C^0} &\leq \sup_l \left\|\mathring{M}_{q,\ell} - \mathring{M}_{q,l}\right\|_{C^0(\operatorname{supp}\mathcal{X}_l)} \\
&\leq \frac{1}{\mu_{q+1}}\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q \\
&\leq \eta\delta_{q+2}.
\end{aligned}$$

Applying the transport estimate Lemma D.0.4 then shows that

$$\begin{aligned}
\left\|\mathring{M}_{q,\ell} - \mathring{M}_{q,l}\right\|_{C^1} &\leq \frac{1}{\mu_{q+1}}\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q\ell \\
&= \delta_{q+1}^{\frac{3}{4}}\delta_q^{\frac{1}{4}}\lambda_q^{\frac{1}{4}}\lambda_{q+1}^{\frac{3}{4}} \\
&\leq \delta_{q+2}\lambda_{q+1},
\end{aligned}$$

providing the desired C^1 bound after recalling that $\partial_z L_{q+1}$ is small.

Moving now to the material derivative, we have that

$$\begin{aligned}
D_{t,q} M_{O,low} &= D_{t,q} \mathcal{D} \mathbb{P}^{\overline{\nabla}} \overline{\nabla} \cdot \Bigg(L_{q+1}^2 \sum_{k+k'=0} \mathcal{X}_l^2 \left[\mathbb{P}_{q+1,k}^{\nabla}, a_{kl} e^{i\lambda_{q+1}(\Phi_l - x)\cdot k} \right] (e^{i\lambda_{q+1} k\cdot x} ik) \otimes \overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp} (\overline{w_{k'l'}}^{\perp}) \\
&\qquad + L_{q+1}^2 \sum_{k+k'=0} \mathcal{X}_l^2 w_{kl} \otimes \left[\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}, a_{k'l'} e^{i\lambda_{q+1}(\Phi_{l'} - x)\cdot k'} \right] (e^{i\lambda_{q+1} k'\cdot x} i\overline{k}'^{\perp}) \Bigg) \\
&\quad + D_{t,q} \left(\mathring{M}_q - \mathring{M}_{q,\ell} + L_{q+1}^2 \sum_l \mathcal{X}_l^2 \left(\mathring{M}_{q,\ell} - \mathring{M}_{q,l} \right) \right) \\
&=: D_{t,q} \mathcal{D} \mathbb{P}^{\overline{\nabla}} \overline{\nabla} \cdot \Omega + D_{t,q} \left(\mathring{M}_q - \mathring{M}_{q,\ell} + L_{q+1}^2 \sum_l \mathcal{X}_l^2 \left(\mathring{M}_{q,\ell} - \mathring{M}_{q,l} \right) \right).
\end{aligned}$$

The second and third terms are the easiest to analyze and we dispense with it first. Since $D_{t,q} L_{q+1}^2 = D_{t,q} \mathring{M}_{q,l} = 0$, we can write that

$$\begin{aligned}
\left\| D_{t,q} \left(L_{q+1}^2 \sum_l \mathcal{X}_l^2 \left(\mathring{M}_{q,\ell} - \mathring{M}_{q,l} \right) \right) \right\|_{C^0} &\leq \left\| D_{t,q} \mathcal{X}_l^2 \right\|_{C^0} \left\| \mathring{M}_q - \mathring{M}_{q,l} \right\|_{C^0} + \left\| D_{t,q} \mathring{M}_{q,\ell} \right\|_{C^0} \\
&\leq \delta_{q+2} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}
\end{aligned}$$

after applying the previous estimate on $D_{t,q} \mathring{M}_{q,\ell}$. In addition, we have that

$$\left\| D_{t,q} \left(\mathring{M}_q - \mathring{M}_{q,\ell} \right) \right\|_{C^0} \leq \delta_{q+2} \delta_{q+1}^{\frac{1}{2}}$$

after applying the inductive assumption and the estimate on $D_{t,q} \mathring{M}_{q,\ell}$.

The first step towards estimating the other term is to estimate the commutator of

$D_{t,q}$ and $\mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot$ applied to Ω using Proposition D.0.6. We can write

$$
\begin{aligned}
&\left\|\left[D_{t,q}, \mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot\right](\Omega)\right\|_{C^0} \leq \|\nabla\Psi_q\|_{C^{1+\alpha}}\|\Omega\|_{C^\alpha} \\
&\quad\leq \|\nabla\Psi_q\|_{C^{1+\alpha}}\Bigg(\left\|\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik)\right\|_{C^\alpha}\left\|\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}(\overline{w_{k'l'}}^{\perp})\right\|_{C^0} \\
&\qquad+\left\|\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik)\right\|_{C^0}\left\|\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}(\overline{w_{k'l'}}^{\perp})\right\|_{C^\alpha} \\
&\qquad+\left\|\left[\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik')\right\|_{C^\alpha}\|w_{kl}\|_{C^0} \\
&\qquad+\left\|\left[\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik')\right\|_{C^0}\|w_{kl}\|_{C^\alpha}\Bigg) \\
&\quad\lesssim \delta_q^{\frac{1}{2}}\lambda_q^{1+\alpha}\delta_{q+1}\lambda_{q+1}^{\alpha} \\
&\quad\leq \delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}
\end{aligned}
$$

if α is small enough. Therefore, it remains to estimate

$$
\begin{aligned}
&\mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot D_{t,q}\left(L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik)\otimes\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}(\overline{w_{k'l'}}^{\perp})\right) \\
&\qquad+\mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot D_{t,q}\left(L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2 w_{kl}\otimes\left[\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}, a_{k'l'}e^{i\lambda_{q+1}(\Phi_{l'}-x)\cdot k'}\right](e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp})\right).
\end{aligned}
$$

We first simplify the above expression by noticing that

$$
\begin{aligned}
&D_{t,q}\Bigg(L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2\left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l-x)\cdot k}\right](e^{i\lambda_{q+1}k\cdot x}ik)\otimes\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}(\overline{w_{k'l'}}^{\perp}) \\
&\qquad+L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2 w_{kl}\otimes\left[\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}, a_{k'l'}e^{i\lambda_{q+1}(\Phi_{l'}-x)\cdot k'}\right](e^{i\lambda_{q+1}k'\cdot x}i\overline{k}'^{\perp})\Bigg) \\
&\quad= D_{t,q}\left(L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2\left(\mathbb{P}^{\nabla}_{q+1,k}(w_{kl})\otimes\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)-w_{kl}\otimes\overline{w_{k'l'}}^{\perp}\right)\right) \\
&\quad= L_{q+1}^2\sum_{k+k'=0}D_{t,q}\left(\mathcal{X}_l^2\right)\left(\mathbb{P}^{\nabla}_{q+1,k}(w_{kl})\otimes\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)-w_{kl}\otimes\overline{w_{k'l'}}^{\perp}\right) \\
&\qquad+L_{q+1}^2\sum_{k+k'=0}\mathcal{X}_l^2\left(D_{t,q}\left(\mathbb{P}^{\nabla}_{q+1,k}(w_{kl})\otimes\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right)\right).
\end{aligned}
$$

Notice that the terms with the projection operators $\mathbb{P}^{\nabla}_{q+1,k}$ and $\overline{\mathbb{P}^{\nabla}_{q+1,k'}}^{\perp}$ are supported in an annulus in x and y frequencies, and so the singular integral operator $\mathcal{D}\mathbb{P}^{\overline{\nabla}}\overline{\nabla}\cdot$ is bounded on

L^∞ for these terms by Lemma D.0.2. Then the entire expression is bounded by

$$\begin{aligned}
&\left\|D_{t,q}\left(\mathcal{X}_l^2\right)\right\|_{C^0}\left(\left\|\mathbb{P}_{q+1,k}^{\nabla}\left(w_{kl}\right)\otimes\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right\|_{C^0}+\left\|w_{kl}\otimes w_{k'l'}\right\|_{C^\alpha}\right)\\
&\qquad+\left\|D_{t,q}\left(\mathbb{P}_{q+1,k}^{\nabla}\left(w_{kl}\right)\otimes\overline{\mathbb{P}_{q+1,k'}^{\nabla}}^{\perp}\left(\overline{w_{k'l'}}^{\perp}\right)\right)\right\|_{C^0}\\
&\leq\mu_{q+1}\left(\delta_{q+1}+\delta_{q+1}\lambda_{q+1}^{\alpha}\right)+\mu_{q+1}\delta_{q+1}\\
&\leq\delta_{q+2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q+1},
\end{aligned}$$

finishing the proof. □

5.6 Energy Increment

In this section, we show that the inductive assumptions (5.12) and (5.13) hold with q replaced by $q+1$. The proof follows estimates of the Hamiltonian increment from [8] and is thus split up into a preliminary lemma and subsequent proposition.

Lemma 5.6.1. *If $t\in\operatorname{supp}\mathcal{X}_l$, then*

$$\left|\int_{\mathbb{T}^3}\left(|\nabla\Psi_q(t)|^2-|\nabla\Psi_q\left(\frac{l}{\mu_{q+1}}\right)|^2\right)\right|+\left|e(t)-e\left(\frac{l}{\mu_{q+1}}\right)\right|\leq\frac{\delta_{q+2}}{16}.$$

Furthermore, for $\rho_l\neq 0$

$$|\rho(t)-\rho_l|\leq\frac{\delta_{q+2}}{16}$$

and

$$e(t)-\int_{\mathbb{T}^3}|\nabla\Psi_q(t)|^2\geq\frac{7\delta_{q+2}}{16}.$$

If $\rho_l=0$, then

$$e\left(\frac{l}{\mu_{q+1}}\right)-\int_{\mathbb{T}^3}|\nabla\Psi_q(t)|^2\leq\frac{9\delta_{q+2}}{16}\quad\text{and}\quad\mathring{M}_q(\cdot,t)\equiv 0.$$

Proof. Using that $\nabla\Psi_q$ solves (5.5) and multiplying by $\nabla\Psi_q$ and integrating by parts, we obtain

$$\begin{aligned}\left|\int_{\mathbb{T}^3}\left(|\nabla\Psi_q(t)|^2-\left|\nabla\Psi_q\left(\frac{l}{\mu_{q+1}}\right)\right|^2\right)\right| &= \left|\int_{\mathbb{T}^3}\int_{\frac{l}{\mu_{q+1}}}^t \mathring{M}_q : \overline{\nabla}\nabla\Psi_q\right| \\ &\le \left(t-\frac{l}{\mu_{q+1}}\right)\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q \\ &\lesssim \frac{4}{\mu_{q+1}}\delta_{q+1}\delta_q^{\frac{1}{2}}\lambda_q \\ &\le \frac{\delta_{q+2}}{32}.\end{aligned}$$

The bound

$$\left|e(t)-e\left(\frac{l}{\mu_{q+1}}\right)\right| \lesssim \frac{1}{\mu_{q+1}} \le \frac{\delta_{q+2}}{32}$$

follows from the smoothness of $e(t)$. Summing both estimates, the first claim is shown. The second claim follows from the first and the definition of $\rho(t)$. The final bound follows from the definition of $\rho(t)$, the first bound, and (5.13). □

Proposition 5.6.2. *If $\rho_l \neq 0$ and $t \in \operatorname{supp}\mathcal{X}_l$, then*

$$\frac{\delta_{q+2}}{4} \le e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_{q+1}(t)|^2 \le \frac{3\delta_{q+2}}{4}.$$

If not, however, then

$$e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_{q+1}(t)|^2 \le \frac{9\delta_{q+2}}{16} \quad \text{and} \quad \mathring{M}_{q+1}(\cdot,t)\equiv 0.$$

Proof. Beginning with the case when $\rho_l = 0$ and $t \in \operatorname{supp}\mathcal{X}_l$, we have that $\nabla\mathbb{W}_{q+1}(t) = 0$, which implies that $\mathring{M}_{q+1}(t) = \mathring{M}_q(t) = 0$ and

$$e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_{q+1}(t)|^2 = e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_q(t)|^2 \le \frac{9\delta_{q+2}}{16}.$$

Moving to the case when $\rho_l \neq 0$ and $t \in \operatorname{supp}\mathcal{X}_l$, then by the frequency and spatial support of $\nabla\Psi_q$ and $\nabla\mathbb{W}_{q+1}$, we have that

$$\begin{aligned}e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_{q+1}(t)|^2 &= e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_q(t)|^2 \\ &\quad - \int_{\mathbb{T}^3}|\nabla(L_{q+1}\mathbb{W}_{q+1})(t)|^2 - 2\int_{\mathbb{T}^3}\nabla\Psi_q(t)\cdot\nabla(L_{q+1}\mathbb{W}_{q+1})(t) \\ &= e(t) - \int_{\mathbb{T}^3}|\nabla\Psi_{q+1}(t)|^2 \\ &\quad - \int_{\mathbb{T}^3}|\nabla(L_{q+1}\mathbb{W}_{q+1})(t)|^2 - 2\int_{\mathbb{T}^3}\nabla\Psi_q(t)\cdot\nabla\mathbb{W}_{q+1}(t).\end{aligned}$$

We have to estimate

$$\int_{\mathbb{T}^3} |\nabla(L_{q+1}\mathbb{W}_{q+1})(t)|^2 + 2\int_{\mathbb{T}^3} \nabla\Psi_q(t)\cdot\nabla\mathbb{W}_{q+1}(t) =: I + II.$$

Using (5.6) and the definition of $\mathbb{P}^{\nabla}_{q+1,k}$ to see that $\nabla\Psi_q$ and $\nabla\mathbb{W}_{q+1}$ are supported in disjoint sets in frequency, we see that $II = 0$. Writing out I gives

$$\begin{aligned}\int_{\mathbb{T}^3} |\nabla(L_{q+1}\mathbb{W}_{q+1})(t)|^2 &= \int_{\mathbb{T}^3} L_{q+1}^2\nabla\mathbb{W}_{q+1}(t)\cdot\nabla\mathbb{W}_{q+1}(t)\\ &\quad + 2\int_{\mathbb{T}^3} L_{q+1}\partial_z L_{q+1}\mathbb{W}_{q+1}\partial_z\mathbb{W}_{q+1} + \int_{\mathbb{T}^3}\left(\partial_z L_{q+1}\right)^2\left(\mathbb{W}_{q+1}\right)^2\\ &= I_1 + I_2 + I_3.\end{aligned}$$

We can control I_2 by

$$\left|2\int_{\mathbb{T}^3} L_{q+1}\partial_z L_{q+1}\mathbb{W}_{q+1}\partial_z\mathbb{W}_{q+1}\right| \le l_{q+1}\frac{\delta_{q+1}}{\lambda_{q+1}}$$

and I_3 by

$$\left|\int_{\mathbb{T}^3}\left(\partial_z L_{q+1}\right)^2\left(\mathbb{W}_{q+1}\right)^2\right| \le l_{q+1}^2\frac{\delta_{q+1}}{\lambda_{q+1}^2}.$$

Writing out I_1 gives

$$\begin{aligned}I_1 &= \sum_{kl}\int_{\mathbb{T}^3} L_{q+1}^2\mathbb{P}^{\nabla}_{q+1,k}(\mathcal{X}_l w_{kl})\mathbb{P}^{\nabla}_{q+1,-k}(\mathcal{X}_l w_{-kl})\\ &= \sum_{kl}\int_{\mathbb{T}^3} L_{q+1}^2\mathcal{X}_l^2\Big[w_{kl}\cdot w_{-kl} + \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right]\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\cdot w_{-kl}\\ &\qquad + w_{kl}\cdot\left[\mathbb{P}^{\nabla}_{q+1,-k}, a_{-kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right]\left(e^{-i\lambda_{q+1}k}ik\right)\\ &\qquad + \left[\mathbb{P}^{\nabla}_{q+1,k}, a_{kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right]\left(e^{i\lambda_{q+1}k\cdot x}ik\right)\cdot\left[\mathbb{P}^{\nabla}_{q+1,-k}, a_{-kl}e^{i\lambda_{q+1}(\Phi_l - x)\cdot k}\right]\left(e^{-i\lambda_{q+1}k}ik\right)\Big]\\ &= \sum_{kl}\int_{\mathbb{T}^3} L_{q+1}^2\mathcal{X}_l^2|a_{kl}|^2 + O\left(\delta_{q+1}\lambda_{q+1}^{-\beta}\right)\\ &= \sum_{l}\mathcal{X}_l^2\rho_l\int_{\mathbb{T}^3} L_{q+1}^2 + O\left(\delta_{q+1}\lambda_{q+1}^{-\beta}\right).\end{aligned}$$

after applying the commutator estimate (D.1) and (5.3). Then applying the definition of ρ_l given in (5.15) finishes the proof. □

5.7 Proof of Main Results

Proof of Proposition 5.4.2. We show that each inductive step holds with q replaced by $q+1$. Referring to the statements of Lemma 5.5.2, Lemma 5.5.3, Lemma 5.5.5, and Lemma 5.5.7, we have that $\nabla\Psi_{q+1}$ solves

$$\partial_t \nabla\Psi_{q+1} + \overline{\nabla}^{\perp}\Psi_{q+1} \cdot \overline{\nabla}\nabla\Psi_{q+1} = \mathrm{curl}(Q_{q+1}) + \overline{\nabla} \cdot \mathring{M}_{q+1}$$

where

$$Q_{q+1} = Q_T + Q_N + Q_{high} + Q_{low}, \qquad \mathring{M}_{q+1} = \mathring{M}_T + \mathring{M}_N + \mathring{M}_{high} + \mathring{M}_{low}$$

and thus (5.5) is satisfied. The inductive step (5.6) follows from the frequency support of $\mathbb{W}_{q+1}L_{q+1}$. (5.7)-(5.11) follow directly from the statements of Lemma 5.5.1, Lemma 5.5.2, Lemma 5.5.3, Lemma 5.5.5, and Lemma 5.5.7. Finally, (5.12) and (5.13) follow from Proposition 5.6.2. □

Proof of Proposition 5.4.3. Towards the purpose of constructing solutions to 2D Euler, one first eliminates the inductive assumption (5.7) on the spatial support and defines $L_{q+1} \equiv 1$ for all q. Next, choose the first set of frequency modes to have zero third component. Then it is easy to see that $\mathring{M}_1$ is of the specified block form. Continuing to apply Lemma 5.3.4 by choosing modes with zero third component since the third row of $\mathring{M}_q$ is empty gives immediately that $\partial_z(\Psi_{q+1} - \Psi_q) \equiv 0$, and therefore Ψ depends only on x, y, and t. □

Proof of Theorem 5.1.1. From the estimate $\|w_{kl}\|_{C^1} + \|L_{q+1}\|_{C^1} \leq \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}$, we have that

$$\begin{aligned}\left\|\nabla\left(\nabla\Psi_{q+1} - \nabla\Psi_q\right)\right\|_{C^0} &= \left\|\nabla^2\left(L_{q+1}\mathbb{W}_{q+1})\right)\right\|_{C^0}\\ &\leq \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.\end{aligned}$$

We claim that the time derivative $\partial_t\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)$ satisfies the same bound. Indeed,

$$\begin{aligned}\left\|\partial_t\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)\right\|_{C^0} &= \left\|D_{t,q}\left(L_{q+1}\nabla\mathbb{W}_{q+1}\right)\right\|_{C^0} + \left\|\overline{\nabla}^{\perp}\Psi_q \cdot \overline{\nabla}\nabla\mathbb{W}_{q+1}\right\|_{C^0}\\ &\leq \mu_{q+1}\delta_{q+1}^{\frac{1}{2}} + \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}\\ &\lesssim \delta_{q+1}^{\frac{1}{2}}\lambda_{q+1}.\end{aligned}$$

Interpolation then shows that

$$
\begin{aligned}
\|\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)\|_{C^{\zeta}_{x,t}} &\leq \|\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)\|^{1-\zeta}_{C^{0}_{x,t}} \|\nabla\left(L_{q+1}\mathbb{W}_{q+1}\right)\|^{\zeta}_{C^{1}_{x,t}} \\
&\lesssim \left(\delta^{\frac{1}{2}}_{q+1}\right)^{1-\zeta}\left(\delta^{\frac{1}{2}}_{q+1}\right)^{\zeta}\lambda^{\zeta}_{q+1} \\
&= \delta^{\frac{1}{2}}_{q+1}\lambda^{\zeta}_{q+1} \\
&= a^{\left(-\frac{1}{2}+c\zeta\right)b^{q+1}}.
\end{aligned}
$$

By the assumption that $c > \frac{5}{2}$, we have that $-\frac{1}{2}+c\zeta$ is negative provided that $\zeta < \frac{1}{2c} < \frac{1}{5}$. Then $\nabla\Psi_q$ is a convergence sequence in $C^{\zeta}_{t,x}$. The bounds on the pressure follow immediately from (5.11) and interpolation. □

Proof of Theorem 5.1.2. Given that the extra assumption of Proposition 5.4.3 is satisfied at each stage q, every subsequent perturbation $\nabla\mathbb{W}_{q+1}$ can be taken to have zero third component, producing a solution to 2D Euler as desired after repeating the steps of the previous proof. □

Appendices

Appendix A

Notes On Chapter 2

Throughout Chapter 2, we use the notation $L^p(\mathbb{R}^n)$ for the usual Lebesgue spaces. The Hilbert Sobolev spaces (for fractional and integer s) are denoted by $H^s(\mathbb{R}^n)$. The homogeneous Sobolev spaces are denoted $\mathring{H}^s(\mathbb{R}^n)$ and are defined as the space of functions f such that $(-\Delta)^{\frac{s}{2}} f \in L^2$. Equivalently, we can define $\mathring{H}^s(\mathbb{R}^n)$ for $s \in (0,1)$ using the Gagliardo seminorm (see Di Nezza, Palatucci, and Valdinoci [43]). To define $\mathring{H}^{\frac{1}{2}}(\Omega)$ for bounded sets $\Omega \in \mathbb{R}^n$, we shall use the Gagliardo seminorm. Negative Sobolev spaces $H^{-z}(\Omega)$ or $H^{-z}(\mathbb{R}^n)$ for $z \in \mathbb{N}$ are defined as the duals of $H_0^z(\Omega)$ or $H^z(\mathbb{R}^n)$, respectively. We use the notation $\nabla^s f$ to denote the collection of all partial derivatives of order $s \in \mathbb{N}$. We shall make use of the following well-known trace estimate for Sobolev functions.

Lemma A.0.1. *Suppose that* $\nabla u \in L^2(\mathbb{R}^3_+)$. *Then* $u|_{z=z_0} \in \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$ *with the trace estimate* $||u(z_0,\cdot)||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} \leq ||\nabla u||_{L^2(\mathbb{R}^3_+)}$.

Lipschitz spaces and their variants will be referred to frequently throughout this chapter.

Definition A.0.1. *1. For* $\alpha \in (0,1)$, *let* $C^\alpha = \{f : ||f||_{C^\alpha} < \infty\}$, *where*

$$||f||_{C^\alpha} = ||f||_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

Also, the homogeneous space $\mathring{C}^\alpha$ *is defined as*

$$\{f : \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty\}.$$

2. Let $\mathrm{Lip} = \{f : ||f||_{\mathrm{Lip}} < \infty\}$, *where*

$$||f||_{\mathrm{Lip}} = ||f||_{L^\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|}.$$

3. Let the space of log-Lipschitz functions $\mathrm{LL} = \{f : ||f||_{\mathrm{LL}} < \infty\}$*, where*

$$||f||_{\mathrm{LL}} = ||f||_{L^\infty} + \sup_{|x-y|<1, x\neq y} \frac{|f(x) - f(y)|}{|x-y|(1-\log(|x-y|))}.$$

Let us now recall the classical Littlewood-Paley operators and the relevant function spaces, as well as some equivalences (we refer to Grafakos [53] or Bahouri, Chemin, and Danchin [3] for details and proofs). Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwarz class of rapidly decaying smooth functions, and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions. Letting $\mathcal{P}$ denote the space of polynomials, we construct the space $\mathcal{S}'/\mathcal{P}$, i.e., tempered distributions modulo polynomials. We employ the standard dyadic decomposition of $\mathbb{R}^n$, specifically a sequence of smooth functions $\{\Phi_j\}_{j\in\mathbb{Z}}$ such that

$$\operatorname{supp} \hat{\Phi}_j \subset \{\xi \in \mathbb{R}^n : |\xi| \in (2^{j-1}, 2^{j+1})\}$$

and

$$\sum_{j\in\mathbb{Z}} \hat{\Phi}_j(\xi) = \begin{cases} 0 & \text{if} \quad \xi = 0 \\ 1 & \text{if} \quad \xi \in \mathbb{R}^n \setminus \{0\}. \end{cases}$$

For $f \in \mathcal{S}'/\mathcal{P}$ and $j \in \mathbb{Z}$, we define $\Delta_j f = \Phi_j * f$.

Definition A.0.2. *For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the space $\mathring{B}^s_{p,q}$ is defined as*

$$\{f \in \mathcal{S}'/\mathcal{P} : ||f||_{\mathring{B}^s_{p,q}} < \infty\}$$

where the homogeneous Besov norm is defined as the l^p norm of the doubly-infinite sequence of Littlewood-Paley projections:

$$||f||_{\mathring{B}^s_{p,q}} = ||\{2^{js}||\Delta_j f||_{L^q}\}_{j\in\mathbb{Z}}||_{l^p}.$$

In nearly every usage throughout this chapter, the Littlewood-Paley projections and the accompanying Besov norms are in $x = (x_1, x_2)$ only; for clarity and emphasis we will use the notation $\mathring{B}^s_{p,q}(\mathbb{R}^2)$. We record the following Bernstein inequalities (see [3]).

Proposition A.0.2. *1. Let $\mathcal{C}$ be an annulus in $\mathbb{R}^d$, $m \in \mathbb{R}$, and $k = 2\lfloor 1 + \frac{d}{2} \rfloor$. Let σ be a k-times differentiable function on $\mathbb{R}^d \setminus \{0\}$ such that for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, there exists a constant C_α such that*

$$\forall \xi \in \mathbb{R}^d, |D^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{m-|\alpha|}.$$

There exists a constant C, depending only on the constants C_α, such that for any $p \in [1,\infty]$ and any $\lambda > 0$, we have, for any function u in L^p with Fourier transform supported in $\lambda\mathcal{C}$,

$$||(\sigma(\xi)\hat{u}(\xi))^\vee||_{L^p} \leq C\lambda^m ||u||_{L^p}.$$

2. *Let $p \in [1,\infty]$ and $s \in \mathbb{R}$. Then for any $j \in \mathbb{Z}$, there exist constants c_1, c_2 such that*

$$c_1 2^{2j\alpha}||\Delta_j u||_{L^p} \leq ||(-\Delta)^\alpha \Delta_j u||_{L^p} \leq c_2 2^{2j\alpha}||\Delta_j u||_{L^p}$$

We several corollaries in the following proposition.

Proposition A.0.3. *Let $s \in \mathbb{R}$, $p, q \in [1,\infty]$.*

1. *Let $\mathcal{R}_j$ denote the j^{th} Riesz transform with Fourier multiplier $\frac{i\xi_j}{|\xi|}$. Then $\mathcal{R}_j$ is a bounded linear operator from $\mathring{B}^s_{p,q}$ to itself.*

2. *Let α be a multi-index. Then the partial differential operator D^α is bounded from $\mathring{B}^s_{p,q}$ to $\mathring{B}^{s-|\alpha|}_{p,q}$.*

3. *Given $\alpha \in \mathbb{R}$, the operator $(-\Delta)^\alpha$ is bounded from $\mathring{B}^s_{p,q}$ to $\mathring{B}^{s-2\alpha}_{p,q}$.*

4. *For $\alpha \in \mathbb{R}$ and $p \in [1,\infty]$, $||f||_{\mathring{B}^\alpha_{p,\infty}} \leq ||(-\Delta)^\alpha f||_{L^p}$.*

We collect several facts concerning the Besov spaces $\mathring{B}^s_{\infty,\infty}$. For a more detailed discussion as well as proofs, see [53].

Proposition A.0.4. 1. *The space $\mathring{B}^1_{\infty,\infty}$ can be characterized as the space of functions such that*

$$||f||_{\mathring{B}^1_{\infty,\infty}} = \sup_{x,y\in\mathbb{R}^n, y\neq 0} \frac{|f(x+y)+f(x-y)-2f(x)|}{|y|} < \infty$$

with equivalence in norm holding between the difference quotient and Littlewood-Paley characterizations.

2. *For non-integer values of s, the spaces $\mathring{B}^s_{\infty,\infty}$ and $\mathring{C}^s$ are equivalent, with an equivalence in norm (which is not uniform in s).*

3. *For any strictly positive s, the restriction of any function $f \in \mathring{B}^s_{\infty,\infty}(\mathbb{R}^n)$ to any k-dimensional affine subset produces a function in $\mathring{B}^s_{\infty,\infty}(\mathbb{R}^k)$ with $||f||_{\mathring{B}^s_{\infty,\infty}(\mathbb{R}^k)} \leq ||f||_{\mathring{B}^s_{\infty,\infty}(\mathbb{R}^n)}$.*

The following proposition will be used in the isoperimetric lemma in the De Giorgi argument.

Proposition A.0.5. 1. *Suppose that $(-\overline{\Delta})^{-\frac{1}{4}}w \in L^\infty(\mathbb{R}^2)$ and $z \in \mathring{H}^{\frac{1}{2}} \cap L^\infty(\mathbb{R}^2)$ is supported in $B_2(0)$. Then there exists C independent of w, z such that*

$$||wz||_{H^{-2}(\mathbb{R}^2)} \leq C||(-\overline{\Delta})^{-\frac{1}{4}}w||_{L^\infty(\mathbb{R}^2)} \left(||z||_{L^\infty(\mathbb{R}^2)} + ||z||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}\right)$$

2. *Suppose that $z \in L^\infty \cap \mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)$. Then there exists C independent of z such that*

$$||z(-\overline{\Delta})^{\frac{1}{2}}z||_{H^{-2}(\mathbb{R}^2)} \leq C\left(||z||_{L^\infty(\mathbb{R}^2)}||z||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + ||z||^2_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}\right).$$

Proof. 1. Suppose that $g \in H^2(\mathbb{R}^2)$. We first show that $(-\overline{\Delta})^{\frac{1}{4}}(zg) \in L^1(\mathbb{R}^2)$. By the compact support of z, we have that

$$\begin{aligned}
|(-\overline{\Delta})^{\frac{1}{4}}(zg)(x)\mathcal{X}_{\{|x|>3\}}(x)| &= \left|\mathcal{X}_{\{|x|>3\}}(x)P.V.\int_{\mathbb{R}^2} \frac{z(x)g(x)-z(y)g(y)}{|x-y|^{\frac{5}{2}}}\,dy\right| \\
&= \left|\mathcal{X}_{\{|x|>3\}}(x)P.V.\int_{B_2(0)} \frac{-z(y)g(y)}{|x-y|^{\frac{5}{2}}}\,dy\right| \\
&\leq C\mathcal{X}_{\{|x|>3\}}(x)\int_{B_2(0)} \frac{||z||_{L^\infty}||g||_{L^\infty}}{|x|^{\frac{5}{2}}}\,dy \\
&\leq C\mathcal{X}_{\{|x|>3\}}(x)\frac{||z||_{L^\infty}||g||_{L^\infty}}{|x|^{\frac{5}{2}}}
\end{aligned}$$

Integrating in x then gives that

$$||(-\overline{\Delta})^{\frac{1}{4}}(zg)(x)\mathcal{X}_{\{|x|>3\}}(x)||_{L^1(\mathbb{R}^2)} \leq C||z||_{L^\infty}||g||_{L^\infty}$$

In addition, it follows from Hölder's inequality and a short calculation with the Gagliardo seminorm that

$$\begin{aligned}
||(-\overline{\Delta})^{\frac{1}{4}}(zg)(x)\mathcal{X}_{\{|x|\leq 3\}}(x)||_{L^1(\mathbb{R}^2)} &\leq C||(-\overline{\Delta})^{\frac{1}{4}}(zg)(x)||_{L^2(\mathbb{R}^2)} \\
&\leq C\left(||z||_{L^\infty}||g||_{\mathring{H}^{\frac{1}{2}}} + ||g||_{L^\infty}||z||_{\mathring{H}^{\frac{1}{2}}}\right)
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^2} wzg &= \int_{\mathbb{R}^2} (-\overline{\Delta})^{-\frac{1}{4}}(w)(-\overline{\Delta})^{\frac{1}{4}}(zg) \\
&\leq ||(-\overline{\Delta})^{-\frac{1}{4}}w||_{L^\infty(\mathbb{R}^2)}||(-\overline{\Delta})^{\frac{1}{4}}(zg)||_{L^1(\mathbb{R}^2)} \\
&\leq C||(-\overline{\Delta})^{-\frac{1}{4}}w||_{L^\infty}\left(||z||_{L^\infty}||g||_{\mathring{H}^{\frac{1}{2}}} + ||g||_{L^\infty}||z||_{\mathring{H}^{\frac{1}{2}}} + ||z||_{L^\infty}||g||_{L^\infty}\right) \\
&\leq C||(-\overline{\Delta})^{-\frac{1}{4}}w||_{L^\infty}\left(||z||_{L^\infty} + ||z||_{\mathring{H}^{\frac{1}{2}}}\right)||g||_{H^2}
\end{aligned}$$

2. Suppose again that $g \in H^2(\mathbb{R}^2)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{2}}z(x)z(x)g(x)\,dx &= \int_{\mathbb{R}^2} (-\overline{\Delta})^{\frac{1}{4}}(z)(x)(-\overline{\Delta})^{\frac{1}{4}}(zg)(x)\,dx \\
&\leq ||(-\overline{\Delta})^{\frac{1}{4}}z||_{L^2(\mathbb{R}^2)}||(-\overline{\Delta})^{\frac{1}{4}}(zg)||_{L^2(\mathbb{R}^2)} \\
&\leq C||z||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}\left(||z||_{L^\infty(\mathbb{R}^2)}||g||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)} + ||g||_{L^\infty(\mathbb{R}^2)}||z||_{\mathring{H}^{\frac{1}{2}}(\mathbb{R}^2)}\right)
\end{aligned}$$

and the result follows from applying Sobolev embedding to g.

□

We recall the well known fact that the characteristic function $\mathcal{X}_E$ of a bounded, Lebesgue measurable set E belongs to H^s if and only if $s < \frac{1}{2}$ (see Bourgain, Brezis, and Mironescu [6] for a detailed discussion). The following is a corollary which will be necessary to prove the decrease in oscillation in the De Giorgi argument.

Proposition A.0.6. *Let ϕ be a radially symmetric and decreasing, C^∞ bump function such that $0 \leq \phi(x) \leq 1$ for all x, $\phi = 1$ on $B_1(0)$, and $\operatorname{supp}\phi \subset B_2(0)$. Let $r(x)$ be a nonnegative, bounded function such that $r^2(x) \in H^{\frac{1}{2}}(\operatorname{supp}\phi)$. Then if $\{x : 0 < r^2(x) < \frac{1}{4}\phi^2\}$ is empty, either $r = 0$ or $r^2 \geq \frac{1}{4}\phi^2$ on $\operatorname{supp}\phi$.*

To control the L^∞ norm of a function by the $\mathring{B}^0_{\infty,\infty}$ Besov norm and some Sobolev norms, we use the following inequality. The proof follows that of Proposition 2.104 in [3]. See also [19] for the same result.

Proposition A.0.7. *There exists a constant C such that for any $h = \overline{\nabla} H : \mathbb{R}^2 \to \mathbb{R}^2$,*

$$||h||_{L^\infty} \leq C||H||_{L^\infty} + C||h||_{\mathring{B}^0_{\infty,\infty}} \left(1 + \log \frac{||h||_{\mathring{H}^{\frac{3}{2}}}}{||h||_{\mathring{B}^0_{\infty,\infty}}}\right).$$

Proof. Let us set $\Theta(x) = 1 - \sum_{j=0}^{\infty} \Phi_j(x)$ where Φ_j is the function associated to the j^{th} Littlewood-Paley projection. Notice that since $\hat{\Theta}(\xi)$ is compactly supported, we have that

$$||\Theta * h||_{L^\infty} = ||\Theta * \overline{\nabla} H||_{L\infty} = ||\overline{\nabla}\Theta * H||_{L^\infty} \leq C||H||_{L^\infty}$$

In addition, we have that by the characterizations of Besov spaces and Sobolev embedding, for $\epsilon = \frac{1}{2}$,

$$\sup_{j \geq 0} 2^{j\epsilon} ||\Delta_j h||_{L^\infty} \leq C||h||_{\mathring{C}^\epsilon} \leq C||h||_{\mathring{H}^{\frac{3}{2}}}.$$

We therefore have that

$$\begin{aligned} ||h||_{L^\infty} &= \left|\left| \Theta * h + \sum_{j=0}^{\infty} \Delta_j h \right|\right|_{L^\infty} \\ &\leq ||\Theta * h||_{L^\infty} + \sum_{j=0}^{N-1} ||\Delta_j h||_{L^\infty} + \sum_{j=N}^{\infty} 2^{j\epsilon} ||\Delta_j h||_{L^\infty} 2^{-j\epsilon} \\ &\leq C||H||_{L^\infty} + N||h||_{\mathring{B}^0_{\infty,\infty}} + C||h||_{\mathring{H}^{\frac{3}{2}}} \frac{2^{-(N-1)\epsilon}}{2^\epsilon - 1} \end{aligned}$$

and taking

$$N = 1 + \left(\frac{1}{\epsilon} \log_2 \frac{||h||_{\mathring{H}^{\frac{3}{2}}}}{||h||_{\mathring{B}^0_{\infty,\infty}}}\right)$$

finishes the proof. □

We now use the above proof to provide a short justification of the construction of the bump functions γ_k in Lemma 2.3.4. Let γ_k be a smooth bump function compactly supported in $B_{\frac{1}{2}+2^{-k-1}}$, equal to $\frac{1}{2} + 2^{-k-1}$ on $B_{\frac{1}{2}+2^{-k-2}}$, and with $||\overline{\nabla}\gamma_k||_{L^\infty} \leq C2^k$. It is clear that the above argument works also for $h = (-\overline{\Delta})^{\frac{1}{2}} H$, and $H = \gamma_k$. Then using that

$$||(-\overline{\Delta})^{\frac{1}{2}} \gamma_k||_{\mathring{B}^0_{\infty,\infty}} \leq ||\overline{\nabla}\gamma_k||_{\mathring{B}^0_{\infty,\infty}} \leq ||\overline{\nabla}\gamma_k||_{L^\infty} \leq C2^k$$

and

$$||(-\overline{\Delta})^{\frac{1}{2}}\gamma_k||_{\mathring{B}^{\epsilon}_{\infty,\infty}} \leq ||\overline{\nabla}\gamma_k||_{\mathring{B}^{\epsilon}_{\infty,\infty}} \leq ||\overline{\nabla}\gamma_k||_{C^\epsilon} \leq C2^{(1+\epsilon)k},$$

we have

$$\begin{aligned}
||(-\overline{\Delta})^{\frac{1}{2}}\gamma_k||_{L^\infty} &\leq ||\gamma_k||_{L^\infty} + C||(-\overline{\Delta})^{\frac{1}{2}}\gamma_k||_{\mathring{B}^0_{\infty,\infty}} \left(1 + \log \frac{||(-\overline{\Delta})^{\frac{1}{2}}\gamma_k||_{\mathring{B}^{\epsilon}_{\infty,\infty}}}{||(-\overline{\Delta})^{\frac{1}{2}}\gamma_k||_{\mathring{B}^0_{\infty,\infty}}}\right) \\
&\leq 1 + C2^k \left(1 + \log C2^{k(1+\epsilon)}\right) \\
&\leq Ck2^k.
\end{aligned}$$

In order to prove propagation of regularity, we need the classical commutator estimate whose proof may be found in Klainerman and Majda [64]. In our case, the control of $||\nabla f||_{L^\infty}, ||g||_{L^\infty}$ will come from the Besov regularity of f and g and Proposition A.0.7.

Proposition A.0.8. *Assume $f, g \in H^s(\mathbb{R}^n)$. Then for any multi-index α with $|\alpha| = s$, we have*

$$||D^\alpha(fg) - fD^\alpha g||_{L^2} \leq C(s) \left(||\nabla f||_{L^\infty}||\nabla^{(s-1)}g||_{L^2} + ||g||_{L^\infty}||\nabla^s f||_{L^2}\right).$$

We require the following lemmas concerning BMO functions to carry out the De Giorgi argument. Here we use BMO to refer to the space of functions with bounded mean oscillation equipped with the usual norm. The first two lemmas are well-known properties of functions belonging to BMO (see [53]). The third follows from the John-Nirenberg inequality [60]. The fourth follows from the third in conjunction with a generalization of the Cauchy-Lipschitz theorem for $L^1(\mathrm{LL})$ vector fields (see Theorem 3.7 in Chapter 3 of [3]). Integrals with a dash through the center are average values.

Proposition A.0.9.

1. *Let Q denote any cube in $\mathbb{R}^n$. For all $0 < p < \infty$, there exists a finite constant $B_{p,n}$ such that*

$$\sup_Q \left(\fint_Q \left| f - \fint_Q f \right|^p\right)^{\frac{1}{p}} \leq B_{p,n}||f||_{\mathrm{BMO}}.$$

2. *Let B_1, B_2 be two balls in $\mathbb{R}^n$ such that there exists A such that*

$$A^{-1}\mathrm{diam}(B_2) \leq \mathrm{diam}(B_1) \leq A\,\mathrm{diam}(B_2)$$

and

$$\operatorname{dist}(B_1, B_2) \le A \operatorname{diam}(B_1).$$

Then there exists a constant $C(A)$ such that for any $u \in \mathrm{BMO}$

$$\left| \fint_{B_1} u - \fint_{B_2} u \right| \le C(A) ||u||_{\mathrm{BMO}}.$$

3. *Let $u \in \mathrm{BMO}$ and satisfy*

$$\sup_{x \in \mathbb{R}^n} \fint_{B_1(x)} u(y)\, dy < \infty$$

and define

$$f(x) = \fint_{B_1(x)} u(y)\, dy.$$

Then $f(x)$ is log-Lipschitz (LL) *in x.*

4. *Let $u(t,x) : [-2,0] \times \mathbb{R}^2 \to \mathbb{R}$ belong to $L^\infty([-2,0]; \mathrm{BMO}(\mathbb{R}^2)) \cap L^\infty([-2,0]; L^2(\mathbb{R}^2))$. Then the following ordinary differential equation has a unique Lipschitz solution which satisfies the ODE almost everywhere in time.*

$$\begin{cases} \dot{\Gamma}(t) = \fint_{B_1(\Gamma(t))} u(t,y)\, dy \\ \Gamma(0) = 0 \end{cases}$$

Finally, we include a short remark on the application of the projection operator $\mathbb{P}_g rad$ to the viscous system. Details and proofs comprise an important part of Chapter 3, and so we omit them here. To adjust their application to the viscous system, we can simply treat the term $\overline{\Delta}\Psi|_{z=0}$ as a forcing term to obtain the following.

Proposition A.0.10. *Let Ψ be a smooth solution to* $(\mathcal{QG})$. *Then if F solves the Neumann problem with $\Delta F = 0$ and $\partial_\nu F = \overline{\Delta}\Psi|_{z=0}$, $\nabla\Psi$ satisfies the following equation, which we shall refer to as (QG_∇):*

$$\partial_t(\nabla\Psi) + \mathbb{P}_\nabla(\overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\nabla\Psi)) = \nabla F.$$

Appendix B

Notes On Chapter 3

In this chapter, we denoted the homogeneous Sobolev spaces by

$$\mathring{W}^{1,r}(\mathbb{R}^3_+) := \{u \in \mathcal{D}'(\mathbb{R}^3_+) | \nabla u \in L^r(\mathbb{R}^3_+)\}$$

with norm

$$\|u\|_{\mathring{W}^{1,r}(\mathbb{R}^3_+)} = \|\nabla u\|_{L^r(\mathbb{R}^3_+)}.$$

Strictly speaking, for the norm to be well-defined and for the following inequality to hold, we consider equivalence classes of distributions which differ by an additive constant. Let us recall the classical Escobar inequality for the half-space $\mathbb{R}^3_+$ [47].

Lemma B.0.1. *Suppose that $q \in [1, 3)$, and $u \in \mathring{W}^{1,q}(\mathbb{R}^3_+)$. Then*

$$\|u|_{z=0}\|_{L^{\frac{2q}{3-q}}(\mathbb{R}^2)} \leq C(q)\|u\|_{\mathring{W}^{1,q}(\mathbb{R}^3_+)}$$

Let us restate and prove the elliptic estimates from Chapter 3.

Lemma 3.2.3. *Given $f \in L^q(\mathbb{R}^3_+)$ for $q \in (1, 3]$, there exists a unique $u \in \mathring{W}^{1,\frac{3q}{3-q}}(\mathbb{R}^3_+)$ ($\nabla u \in BMO$ if $q = 3$) such that*

$$\begin{cases} -\Delta u = f & z > 0 \\ \partial_\nu u = 0 & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3q}{3-q}}(\mathbb{R}^3_+)} \leq C(q)\|f\|_{L^q(\mathbb{R}^3_+)}, \qquad q < 3$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}^3_+)} \leq C(q)\|f\|_{L^q(\mathbb{R}^3_+)}, \qquad q = 3.$$

Proof. Let us begin with the case $q = 3$. Applying the operator whose symbol is $\frac{i\xi}{|\xi|^2}$ (we ignore constants coming from the Fourier transform) to

$$f_E(z,x) = \begin{cases} f(z,x) & z > 0 \\ 0 & z \leq 0 \end{cases}$$

gives a curl free vector field in $BMO(\mathbb{R}^3)$ which is in fact the gradient of a function u_E (see, for example, Temam [92]). Then applying the same operator to

$$f_{E,r}(z,x) = \begin{cases} 0 & z > 0 \\ f(-z,x) & z \leq 0 \end{cases}$$

yields a vector field in $BMO(\mathbb{R}^3)$ which is again the gradient of a function $u_{E,r}$. Putting $u = u_E + u_{E,r}$, it is clear that $-\Delta u = f$ in $\mathbb{R}^3_+$ and

$$\partial_\nu u = \partial_\nu u_E + \partial_\nu u_{E,r} = \partial_\nu u_E - \partial_\nu u_E = 0.$$

The bound follows from the boundedness of the multiplier operator from $L^3(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$.

We use the generalized Lax-Milgram theorem for Banach spaces (see for example Theorem 8.10 in the text of Arbogast and Bona [1]) to show the existence as well as the bound for $q < 3$. Define $X := \mathring{W}^{1,\frac{a}{a-1}}(\mathbb{R}^3_+)$ and $Y := \mathring{W}^{1,a}(\mathbb{R}^3_+)$. Define $B : X \times Y \to \mathbb{R}$ by

$$B(u,v) = \int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v$$

and $F(v) : Y \to \mathbb{R}$ by

$$F(v) = \int_{\mathbb{R}^3_+} vf.$$

Choosing $a = \frac{3q}{4q-3}$ gives that $v \in L^{\frac{q}{q-1}}(\mathbb{R}^3_+)$ by Sobolev embedding, and thus F is well-defined and continuous. Continuity of B follows from Hölder's inequality. We must show B to be non-degenerate, i.e.

$$\sup_{u \in X} B(u,v) > 0 \quad \forall v \in Y$$

and coercive, i.e.

$$\inf_{\substack{u \in X \\ \|u\|=1}} \sup_{\substack{v \in Y \\ \|v\|=1}} B(u,v) \geq \gamma > 0.$$

To show coercivity, we begin by fixing $u \in X$ with $\|u\|_{\mathring{W}^{1,\frac{a}{a-1}}} = 1$. The ideal choice for ∇v would be $\nabla u |\nabla u|^{\frac{a}{a-1}-2}$. Of course, this may not be the gradient of a function. Therefore, let us define the operator $\mathbb{P}_\nabla$ for Schwartz vector fields $s : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\begin{aligned}\widehat{\mathbb{P}_\nabla(s)}(\xi) &= \left(\frac{\langle \hat{s}(\xi), \xi \rangle}{|\xi|^2} \xi \right) \\ &= \left(\sum_{i=1}^{3} \hat{s}_i(\xi) \frac{\xi_i \xi_j}{|\xi|^2} \right), \qquad j = 1, 2, 3.\end{aligned}$$

Recalling that the symbol for the j^{th} Riesz transform $\mathcal{R}_j$ is $-\frac{i\xi_j}{|\xi|}$, $\mathbb{P}_\nabla$ is a linear combination of compositions of Riesz transforms. We then extend $\mathbb{P}_\nabla$ by density as a bounded operator from $\left(L^r(\mathbb{R}^3)\right)^3$ to itself for all $r \in (1, \infty)$. In addition, for s_1 scalar valued, s_2 vector valued Schwarz functions, examining the symbol of $\mathbb{P}_\nabla$ shows that

$$\langle \nabla s_1, \mathbb{P}_\nabla s_2 \rangle = \langle \nabla s_1, s_2 \rangle.$$

Continuity of the operator ensures that this property remains true for vector fields in X and Y. We define $u_E(z, x) = u(|z|, x)$ to be the symmetric extension of u over the plane $z = 0$. With this definition,

$$\partial_z u_E(z, x) = -\partial_z u_E(-z, x) \tag{B.1}$$

and

$$\overline{\nabla} u_E(z, x) = \overline{\nabla} u_E(-z, x). \tag{B.2}$$

We apply $\mathbb{P}_\nabla$ to the extended vector field $\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2}$. Using the symmetry and anti-symmetry of the Riesz transforms and $\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2}$ with respect to reflection over the plane $z = 0$, it is simple to check that

$$\partial_z \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right)(z, x) = -\partial_z \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right)(-z, x) \tag{B.3}$$

and

$$\overline{\nabla} \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right)(z, x) = \overline{\nabla} \mathbb{P}_\nabla \left(\nabla u_E |\nabla u_E|^{\frac{a}{a-1}-2} \right)(-z, x). \tag{B.4}$$

We set $\nabla v(z,x) = \frac{1}{\|\mathbb{P}_\nabla\|}\mathbb{P}_\nabla(\nabla u_E|\nabla u_E|^{\frac{a}{a-1}-2})|_{z\geq 0}$. By direct computation, $\|\nabla v\|_{L^a(\mathbb{R}^3_+)} \leq 1$, and using (B.1), (B.2), (B.3), and (B.4) gives that

$$\begin{aligned}
\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v &= \frac{1}{2\|\mathbb{P}_\nabla\|}\int_{\mathbb{R}^3} \nabla u_E \cdot \mathbb{P}_\nabla(\nabla u_E|\nabla u_E|^{\frac{a}{a-1}-2}) \\
&= \frac{1}{2\|\mathbb{P}_\nabla\|}\int_{\mathbb{R}^3} |\nabla u_E|^{\frac{a}{a-1}} \\
&= \frac{1}{\|\mathbb{P}_\nabla\|}.
\end{aligned}$$

Thus the coercivity is shown with $\gamma = \frac{1}{\|\mathbb{P}_\nabla\|}$. Non-degeneracy follows from switching u and v and repeating the argument. Therefore, the conditions of Lax-Milgram are met, and we have the existence of a solution u to the variational problem, as well as the gradient bound on u in terms of f. Then, taking v to be compactly supported in $\mathbb{R}^3_+$ shows that $-\Delta u = f$ in the sense of distributions. Now, taking $v \in \mathcal{D}(\mathbb{R}^3)$ shows that $\partial_\nu u$ is well defined as a distribution by

$$\int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v + v\Delta u =: \int_{\mathbb{R}^2} v\partial_\nu u$$

and is equal to zero. $\square$

Lemma 3.2.4. *Given $g \in L^p(\mathbb{R}^2)$ for $p \in (1,\infty]$, there exists $u \in \mathring{W}^{1,\frac{3p}{2}}_\Delta(\mathbb{R}^3_+)$ solving*

$$\begin{cases} \Delta u = 0 & z > 0 \\ \partial_\nu u = g & z = 0 \end{cases}$$

with

$$\|\nabla u\|_{L^{\frac{3p}{2}}(\mathbb{R}^3_+)} \leq C(p)\|g\|_{L^p(\mathbb{R}^2)}, \qquad p < \infty$$

or

$$\|\nabla u\|_{BMO(\mathbb{R}^3_+)} \leq C(p)\|g\|_{L^p(\mathbb{R}^2)}, \qquad p = \infty.$$

Proof. Let us begin with the case $p = \infty$. Applying the Poisson kernel $\mathcal{P}(z,x)$ to $g(x)$ gives a harmonic function in $\mathbb{R}^3_+$. Considering the vector field

$$v(z,x) = -\left(\mathcal{P}(z,\cdot) * g(\cdot)(x), \mathcal{R}_1\mathcal{P}(z,\cdot) * g(\cdot)(x), \mathcal{R}_2\mathcal{P}(z,\cdot) * g(\cdot)(x)\right),$$

it is clear that v is curl free and is thus the gradient of a harmonic function u with $\partial_\nu u = g$. The bound follows from noting that the Riesz transforms are bounded from $L^\infty(\mathbb{R}^2)$ to $BMO(\mathbb{R}^2)$ and $\|\mathcal{P}(z,\cdot) * g(\cdot)(x)\|_{L^\infty(\mathbb{R}^2)} \leq \|g(x)\|_{L^\infty(\mathbb{R}^2)}$ for all z.

We use again the Lax-Milgram theorem for $p < \infty$. Define $X := \mathring{W}_\Delta^{1,\frac{3p}{2}}(\mathbb{R}^3_+)$ and $Y := \mathring{W}^{1,\frac{3p}{3p-2}}(\mathbb{R}^3_+)$. Let $B : X \times Y \to \mathbb{R}$ be defined by

$$B(u,v) = \int_{\mathbb{R}^3_+} \nabla u \cdot \nabla v$$

and $F : Y \to \mathbb{R}$ be defined by

$$F(v) = \int_{\mathbb{R}^2} v|_{z=0} g.$$

By Lemma A.0.1, we have that $v|_{z=0} \in L^{\frac{p}{p-1}}(\mathbb{R}^2)$, and therefore F is well-defined and continuous. Continuity of B follows from Hölder's inequality. As before, we are tasked with showing the coercivity and non-degeneracy of B. Making use of the $\mathbb{P}_\nabla$ operator, the details follow as in the previous lemma and are omitted. The existence of u and the gradient bound in terms of g are provided by the Lax-Milgram theorem. Taking v compactly supported in $\mathbb{R}^3_+$ shows that indeed $\Delta u = 0$. We then again have that $\partial_\nu u$ is well-defined as a distribution from integration by parts and satisfies $\partial_\nu u = g$. $\square$

Lemma 3.2.5. *Let $\{g_\epsilon\}_{\epsilon>0}$ be a bounded sequence of functions in $L^p(\mathbb{R}^2)$ for $p > \frac{4}{3}$. Let $u_\epsilon(z,x) : \mathbb{R}^3_+ \to \mathbb{R}$ be the solution to*

$$\begin{cases} \Delta u_\epsilon = 0 & z > 0 \\ \partial_\nu u_\epsilon = g_\epsilon & z = 0 \end{cases}$$

Then there exists u such that up to a subsequence, ∇u_ϵ converges strongly to ∇u in $L^2\left((0,R) \times B_R(0)\right)$ for all $R > 0$.

Proof. Fix $R > 0$. We first extract a subsequence which we shall continue to call $\{g_\epsilon\}$ in an abuse of notation that converges weakly-* to g in $L^p(\mathbb{R}^2)$. Applying Lemma 3.2.4 to g_ϵ gives that u_ϵ converges weakly-* to u in $\mathring{W}^{1,\frac{3p}{2}}(\mathbb{R}^3_+)$, where u solves the Laplace equation with Neumann data g. Because $p > \frac{4}{3}$, we have that $\frac{3p}{2} > 2$, and therefore $\{\nabla u_\epsilon\}$ is a weakly-* convergent sequence in $L^2\left((0,R) \times B_R(0)\right)$. Note that ∇u_ϵ is harmonic for all ϵ and thus the harmonic extension satisfies for fixed z that

$$\nabla u_\epsilon(z,x) = -\left(\mathcal{P}(z,\cdot) * g_\epsilon(\cdot)(x), \mathcal{R}_1\mathcal{P}(z,\cdot) * g_\epsilon(\cdot)(x), \mathcal{R}_2\mathcal{P}(z,\cdot) * g_\epsilon(\cdot)(x)\right),$$

and similarly for u. Furthermore, by the smoothness and decay at infinity of the Poisson kernel $\mathcal{P}$ away from the boundary $z = 0$, $\mathcal{P} * g_\epsilon$ and $\mathcal{P} * g$ belong to $W^{k,\frac{3p}{2}}((\delta, \infty) \times \mathbb{R}^2)$ for any $k \in \mathbb{N}$ and fixed $\delta > 0$. Taking the Riesz transform shows that the same holds for $\mathcal{R}(\mathcal{P} * g_\epsilon)$ and $\mathcal{R}(\mathcal{P} * g)$. Then by the Rellich-Kondrachov theorem, ∇u_ϵ converges strongly to ∇u up to a subsequence in $L^2((\delta, R) \times B_R(0))$. Thus fixing $0 < \delta < R$, we can write

$$\begin{aligned}
\limsup_{\epsilon \to 0} \int_0^R \int_{B_R(0)} |\nabla u_\epsilon(z,x) - \nabla u(z,x)|^2 \, dx \, dz \\
&= \limsup_{\epsilon \to 0} \left(\int_0^\delta \int_{B_R(0)} |\nabla u_\epsilon(z,x) - \nabla u(z,x)|^2 \, dx \, dz \right. \\
&\qquad \left. + \int_\delta^R \int_{B_R(0)} |\nabla u_\epsilon(z,x) - \nabla u(z,x)|^2 \, dx \, dz \right) \\
&\le \sup_{\epsilon > 0} \int_0^\delta \int_{B_R(0)} |\nabla u_\epsilon(z,x) - \nabla u(z,x)|^2 \, dx \, dz \\
&\le \sup_{\epsilon > 0} \|\nabla u_\epsilon - \nabla u\|^2_{L^{\frac{3p}{2}}(\mathbb{R}^3_+)} \|\mathcal{X}_{\{[0,\delta] \times B_R(0)\}}\|_{L^{\frac{3p}{3p-4}}(\mathbb{R}^3_+)} \\
&\le C \left(R^2 \delta\right)^{\frac{3p-4}{3p}}
\end{aligned}$$

after applying the uniform bound on g_ϵ in $L^p(\mathbb{R}^2)$ and Hölder's inequality. Considering that $p > \frac{4}{3}$ and R is fixed, the final expression approaches zero as δ decreases to zero. Diagonalizing the subsequence u_ϵ over $R \in \mathbb{N}$ finishes the proof. $\square$

We recall the Hodge decomposition from Vasseur and Puel [82], with an additional higher regularity bound which will be useful.

Lemma B.0.2 (Hodge Decomposition). *Let $v \in H^3(\mathbb{R}^3_+)$. Then there exists a unique decomposition*

$$v = \nabla w + \operatorname{curl} u, \qquad \nabla w, \operatorname{curl} u \in L^2(\mathbb{R}^3_+)$$

satisfying

$$\int_{\mathbb{R}^3_+} \nabla \phi \cdot v = \int_{\mathbb{R}^3_+} \nabla \phi \cdot \nabla w$$

for any $\nabla \phi \in L^2(\mathbb{R}^3_+)$. In addition, we have the following higher regularity bound:

$$\|\nabla w\|_{H^3(\mathbb{R}^3_+)} \lesssim \|v\|_{H^3(\mathbb{R}^3_+)}.$$

Finally, if the support of v is compact, then $\nabla^2 w \in L^{1+\delta}(\mathbb{R}^3_+)$ for any $\delta > 0$.

Proof. Proposition 3.2 from Vasseur and Puel's work [82] shows the existence of the unique decomposition

$$v = \nabla w + \operatorname{curl} u$$

given $v \in L^2(\mathbb{R}^3_+)$ with the desired orthogonality condition in $L^2(\mathbb{R}^3_+)$. Since $v \in H^3(\mathbb{R}^3_+)$, in particular $v \in L^2(\mathbb{R}^3_+)$, and we can apply their result to conclude the existence of a unique ∇w, $\operatorname{curl} u$ belonging to $L^2(\mathbb{R}^3_+)$ and satisfying

$$\int_{\mathbb{R}^3_+} \nabla\phi \cdot v = \int_{\mathbb{R}^3_+} \nabla\phi \cdot \nabla w.$$

We now show the higher regularity bound $\|\nabla w\|_{H^3} \lesssim \|v\|_{H^3(\mathbb{R}^3_+)}$. The proof utilizes the classical Nirenberg difference quotients.

Let the difference quotient operator T_h for the chosen direction $x = (x_1, x_2) \in \mathbb{R}^2$ be defined by

$$T_h(f)(z', x') := \frac{f(z', x' + hx) - f(z', x')}{h}$$

and let ∂_x be the corresponding partial differential operator. Then all quantities in the following expression are well-defined and we can write

$$\begin{aligned}\int_{\mathbb{R}^3_+} \nabla(T_h w) \cdot \nabla(T_h w) &= -\int_{\mathbb{R}^3_+} \nabla(T_{-h} T_h w) \cdot \nabla w \\ &= -\int_{\mathbb{R}^3_+} \nabla(T_{-h} T_h w) \cdot v \\ &= \int_{\mathbb{R}^3_+} \nabla(T_h w) \cdot (T_h v).\end{aligned}$$

Applying Cauchy's inequality, we conclude

$$\|\nabla(T_h w)\|_{L^2(\mathbb{R}^3_+)} \lesssim \|v\|_{H^1(\mathbb{R}^3_+)}$$

with a bound uniform in h. Passing to a limit as $h \to 0$ shows that then $\nabla(\partial_x w) \in L^2(\mathbb{R}^3_+)$; to show that $\partial_{zz} w \in L^2(\mathbb{R}^3_+)$, we observe that $\partial_{zz} w = \Delta w - \overline{\Delta} w = \nabla \cdot v - \overline{\Delta} w$. Therefore

$$\|\nabla w\|_{H^1} \lesssim \|v\|_{H^1}.$$

For the H^2 bound, we can first write that

$$\begin{aligned}\int_{\mathbb{R}^3_+} \nabla(T_hT_hw)\cdot\nabla(T_hT_hw) &= \int_{\mathbb{R}^3_+} \nabla(T_{-h}T_{-h}T_hT_hw)\cdot\nabla w \\ &= \int_{\mathbb{R}^3_+} \nabla(T_{-h}T_{-h}T_hT_hw)\cdot v \\ &= \int_{\mathbb{R}^3_+} \nabla(T_hT_hw)\cdot(T_hT_hv).\end{aligned}$$

From here we conclude as before that $\|\nabla(\partial_{xx}w)\|_{L^2} \lesssim \|v\|_{H^2}$. Since $\partial_{zzx} = \partial_x(\Delta - \overline{\Delta})$ and $\Delta w = \nabla\cdot v$, we have that $\partial_{zzx}w \in L^2$. In addition, since $\partial_{zzz} = \partial_z(\Delta - \overline{\Delta})$, we have that $\partial_{zzz}w \in L^2$, and therefore

$$\|\nabla w\|_{H^2} \lesssim \|v\|_{H^2}.$$

For the H^3 bound, we can argue as above to conclude that $\|\nabla(\partial_{xxx}w)\| \lesssim \|v\|_{H^3}$. The full bound then follows from the identities

$$\partial_{zzxx} = \partial_{xx}(\Delta - \overline{\Delta}), \qquad \partial_{zzzx} = \partial_{zx}(\Delta - \overline{\Delta}), \qquad \partial_{zzzz} = \partial_{zz}(\Delta - \overline{\Delta}).$$

It remains to show the $L^{1+\delta}$ bound on $\nabla^2 w$ in the case that $\operatorname{supp} v$ is compact. When $\operatorname{supp} v$ is compact, $\Delta w = \nabla\cdot v$ and $\partial_\nu w = v\cdot\nu$ are therefore both compactly supported and L^1. By the elliptic bounds in Lemma 3.2.3 and Lemma 3.2.4, $\nabla w \in L^{\frac{3}{2}+\delta}(\mathbb{R}^3_+)$ for any $\delta > 0$. By Sobolev embedding, $w \in L^{3+\delta}(\mathbb{R}^3_+)$ for any $\delta > 0$. Classical estimates for harmonic functions then give that for $\alpha \in \mathbb{R}^3_+$ sufficiently large (far outside the support of v),

$$\begin{aligned}|\nabla^2 w(\alpha)| &\lesssim \frac{1}{|\alpha|^5}\|w\|_{L^1(B(\alpha,\frac{|\alpha|}{2}))} \\ &\lesssim \frac{1}{|\alpha|^5}\|w\|_{L^{3+\delta}(\mathbb{R}^3_+)}|\alpha|^{3\left(\frac{2+\delta}{3+\delta}\right)}.\end{aligned}$$

Thus $\nabla^2 w$ decays at a rate of $\frac{1}{|\alpha|^{3-\delta}}$ in $\mathbb{R}^3_+$ for any $\delta > 0$, showing that $\nabla^2 w \in L^{1+\delta}(\mathbb{R}^3_+)$ for any $\delta > 0$. $\square$

Finally, we provide an outline of the proof of the existence theorem for the regularized system with forcing terms following the arguments of Chapter 2.

Theorem 3.2.1. *Consider the regularized system* $(QG)_\epsilon$

$$\begin{cases} \partial_t(\Delta\Psi_\epsilon) + \overline{\nabla}^\perp\Psi_\epsilon \cdot \overline{\nabla}(\Delta\Psi_\epsilon) = f_{L,\epsilon} & t>0, \quad z>0, \quad x=(x_1,x_2)\in\mathbb{R}^2 \\ \partial_t(\partial_\nu\Psi_\epsilon) + \overline{\nabla}^\perp\Psi_\epsilon \cdot \overline{\nabla}(\partial_\nu\Psi_\epsilon) = f_{\nu,\epsilon} - \epsilon(-\overline{\Delta})^{\frac{1}{2}}(\partial_\nu\Psi_\epsilon) & t>0, \quad z=0, \quad x=(x_1,x_2)\in\mathbb{R}^2 \end{cases}$$

supplied with initial data $\Delta\Psi_{0,\epsilon}$*,* $\partial_\nu\Psi_{0,\epsilon}$ *which are* C^∞ *and compactly supported. Suppose that* $f_{L,\epsilon} \in L^1\left([0,T]; L^1 \cap L^q(\mathbb{R}^3_+)\right) \cap L^\infty([0,T]; C^k(\mathbb{R}^3_+))$*,* $f_{\nu,\epsilon} \in L^1\left([0,T]; L^1 \cap L^p(\mathbb{R}^2)\right) \cap L^\infty([0,T]; C^k(\mathbb{R}^2))$ *for all* $T>0$*,* $k\in\mathbb{N}$ *and that for each time,* $f_{L,\epsilon}$ *and* $f_{\nu,\epsilon}$ *have spatial support contained in* $[-\frac{5}{\epsilon},\frac{5}{\epsilon}]^3$ *and* $[-\frac{5}{\epsilon},\frac{5}{\epsilon}]^2$*, respectively. Then there exists a unique, global in time classical solution* $\nabla\Psi_\epsilon$ *and a constant* C *independent of* ϵ *such that* $\nabla\Psi_\epsilon$ *satisfies the energy estimates for* $t\in[0,T]$

1. $\|\Delta\Psi_\epsilon(t)\|_{L^q} \le C(\|f_{L,\epsilon}\|_{L^1([0,T];L^q)} + \|\Delta\Psi_{0,\epsilon}\|_{L^q})$

2. $\|\partial_\nu\Psi_\epsilon(t)\|_{L^p} \le C(\|f_{\nu,\epsilon}\|_{L^1([0,T];L^p)} + \|\partial_\nu\Psi_{0,\epsilon}\|_{L^p})$

3. $\|\nabla\Psi_\epsilon(t)\|_{L^{\frac{3q}{3-q}}+L^{\frac{3p}{2}}} \le C(\|f_{L,\epsilon}\|_{L^1([0,T];L^q)} + \|\Delta\Psi_{0,\epsilon}\|_{L^q} + \|f_{\nu,\epsilon}\|_{L^1([0,T];L^p)} + \|\partial_\nu\Psi_{0,\epsilon}\|_{L^p})$

Proof of Theorem 3.2.1. The differences between the setting of Chapter 2 and $(QG)_\epsilon$ are the presence of smooth forcing terms and the replacement of the term $-\overline{\Delta}\Psi_\epsilon$ (which comes from the physical consideration of Ekman layers) with the simplified diffusive term $(-\overline{\Delta})^{\frac{1}{2}}\partial_\nu\Psi_\epsilon$. When $\Delta\Psi \equiv 0$, the two diffusive terms are equal. When Ψ is not harmonic, $(-\overline{\Delta})^{\frac{1}{2}}\partial_\nu\Psi_\epsilon$ is easier to analyze, as it ignores the effect of interior vorticity which appears in the term $\overline{\Delta}\Psi_\epsilon$. Local in time existence of smooth solutions in both [79] and $(QG)_\epsilon$ follows from classical semigroup techniques, such as those formulated by Kato [61]. Then, the proof of global existence is predicated on estimates which show that a sufficient level of regularity of the trajectories of the velocity field $\overline{\nabla}^\perp\Psi_\epsilon$ depends *only* on quantities which are preserved by the evolution of the system. Applying a continuation criterion finishes the proof. Our goal is to provide an outline of the simple changes needed to apply those arguments to $(QG)_\epsilon$.

The main estimates in Chapter 2 show that if

$$\overline{\nabla}^\perp\Psi \in L^\infty([0,T]\times[0,\infty); \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2)) \tag{B.5}$$

then the higher Sobolev norms of $\nabla\Psi$ satisfy a differential inequality on $[0,T]$, showing that the solution can be continued beyond time T. Adding smooth forcing terms $f_{L,\epsilon}, f_{\nu,\epsilon}$ to the right hand side will introduce terms depending on Sobolev norms of $f_{L,\epsilon}, f_{\nu,\epsilon}$ into the differential inequality for $\|\nabla\Psi\|_{H^s}$; as long as the forcing terms are smooth, the argument functions in the same manner as the non-forced case.

The bulk of the argument then consists of showing that the estimate (B.5) is preserved by the evolution of the system and does not blow up in finite time. This is achieved in three main steps. First, the de Giorgi technique is applied to obtain a C^α estimate on $\partial_\nu\Psi$. Second, a bootstrapping argument combining potential theory and Littlewood-Paley techniques shows that $\partial_\nu\Psi \in L^\infty([0,T];\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. Third, it is shown that once $\partial_\nu\Psi \in L^\infty([0,T];\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$, (B.5) must hold. The third step requires no adaptations and we briefly describe it now before moving to the first two. Using the notation for $\Psi_{1,\epsilon}$ and $\Psi_{2,\epsilon}$ as throughout the paper, simple properties of the Riesz transform and the Poisson kernel show that

$$\|\overline{\nabla}^\perp\Psi_{1,\epsilon}\|_{L^\infty([0,T]\times[0,\infty);\mathring{B}^1_{\infty,\infty})} \lesssim \|\partial_\nu\Psi_{1,\epsilon}\|_{L^\infty([0,T];\mathring{B}^1_{\infty,\infty})} = \|\partial_\nu\Psi_\epsilon\|_{L^\infty([0,T];\mathring{B}^1_{\infty,\infty})}.$$

In addition, since $\Delta\Psi_\epsilon = \Delta\Psi_{2,\epsilon}$ solves a transport equation with divergence free drift and smooth forcing, the method of characteristics shows that the $L^\infty(\mathbb{R}^3_+)$ norm of $\Delta\Psi_{2,\epsilon}$ depends only on the initial data and the forcing term $f_{L,\epsilon}$. Then, properties of the Riesz transforms and classical trace estimates for Besov spaces show that $\overline{\nabla}^\perp\Psi_{2,\epsilon} \in L^\infty([0,T]\times[0,\infty);\mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$, and thus (B.5) holds. We move then to the first two steps.

The de Giorgi argument is written for equations with divergence free drift u and forcing f

$$\partial_t\theta + u\cdot\overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = f.$$

Setting $\theta = \partial_\nu\Psi_\epsilon$, $u = \overline{\nabla}^\perp\Psi_\epsilon$, and $f = f_{\nu,\epsilon}$ shows that $(QG)_\epsilon$ falls into this regime. The steps of the De Giorgi argument include a global L^∞ bound, a local L^∞ bound, an isoperimetric lemma, and a decrease in oscillation. Combining each step yields a Hölder modulus of continuity which depends only on $\|\Psi_{0,\epsilon}\|_{H^3(\mathbb{R}^3_+)}$ and certain norms of the forcing $f_{\nu,\epsilon}$. As the initial data and forcing have been regularized, these bounds are satisfied for $(QG)_\epsilon$. To give a flavor of the de Giorgi arguments, we state the global L^∞ bound; the following steps can be stated entirely analogously to the lemmas from Chapter 2.

Lemma B.0.3 (Global L^∞ bound)**.** *For any $M > 0$, there exists $L > 0$ such that the following holds. Let $\theta \in L^\infty([-2,0]; H^{\frac{5}{2}}(\mathbb{R}^2))$ be a solution to*

$$\partial_t \theta + u \cdot \overline{\nabla} \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = f$$

with

$$||\theta||_{L^\infty([-2,0];L^2(\mathbb{R}^2))} + ||(-\overline{\Delta})^{-\frac{1}{4}} f||_{L^\infty([-2,0];C^{\frac{1}{2}}(\mathbb{R}^2))} < M$$

and $\operatorname{div} u = 0$*. Then $\theta(t,x) \leq L$ for $(t,x) \in [-1,0] \times \mathbb{R}^2$.*

Finally, let us describe the bootstrapping argument. The bootstrapping argument is built around the observation that the Poisson kernel is the fundamental solution to the equation

$$\partial_t \theta + (-\overline{\Delta})^{\frac{1}{2}} \theta = 0.$$

The choice of $(-\overline{\Delta})^{\frac{1}{2}} \partial_\nu \Psi_\epsilon$ as the diffusive term ensures that $(QG)_\epsilon$ again falls into this regime. In both Chapter 2 and $(QG)_\epsilon$, two forcing terms appear on the right hand side. The first term in both settings is of the form $u \cdot \overline{\nabla} \theta$ and comes from the nonlinearity. The second term in Chapter 2 comes from the effect of Ψ_2 on the diffusive term $\overline{\Delta} \Psi$, whereas in $(QG)_\epsilon$ it comes from $f_{\nu,\epsilon}$. From Section 2.4 the regularity of the nonlinear term is effectively *additive*. That is, if f and g are C^α and θ_1 solves

$$\partial_t \theta_1 + (-\overline{\Delta})^{\frac{1}{2}} \theta_1 = \overline{\nabla} \cdot (g_1 g_2),$$

the representation formula for θ_1 given by the Poisson kernel gives a $C^{2\alpha}$ estimate on θ_1. Repeating this argument bootstraps the regularity of θ_1 all the way to $C^{1,\alpha}$ for any $\alpha \in (0,1)$. Setting $g_1 = \overline{\nabla}^\perp \Psi_\epsilon$ and $g_2 = \partial_\nu \Psi_\epsilon$ allows us to apply Lemma 4.1 to $(QG)_\epsilon$. Finally, as the forcing term $f_{\nu,\epsilon}$ is smooth in space, the solution to the fractional heat equation

$$\partial_t \theta_2 + (-\overline{\Delta})^{\frac{1}{2}} \theta_2 = f_{\nu,\epsilon}$$

is smooth in space as well. Setting $\theta = \theta_1 + \theta_2$ and combining the two arguments shows that $\theta \in L^\infty([0,T]; \mathring{B}^1_{\infty,\infty}(\mathbb{R}^2))$. We refer again to the discussion in Section 2.4 which precedes Theorem 4.3 for details on the combination of these two steps. The conclusion is that

$\partial_\nu \Psi_\epsilon \in L^\infty([0,T]; \mathring{B}^1_{\infty,\infty})$ with norm depending only on the initial data $\Psi_{0,\epsilon}$ and the forcing terms, concluding the construction of a smooth solution.

The estimate in (1) follows from the method of characteristics. For (2), recall that for $0 < \alpha \leq 2$, $1 \leq p < \infty$ the following inequality holds (see [71] for example):

$$0 \leq \int_{\mathbb{R}^2} \theta |\theta|^{p-2} \Lambda^\alpha \theta.$$

Multiplying by $\partial_\nu \Psi_\epsilon |\partial_\nu \Psi_\epsilon|^{p-2}$, integrating by parts, and applying the inequality with $\theta = \partial_\nu \Psi_\epsilon$ and $\alpha = 1$ then shows (2) for $p < \infty$. The estimate for $p = \infty$ follows after noticing that the initial data $\partial_\nu \Psi_{0,\epsilon}$ and the forcing $f_{\nu,\epsilon}$ are smooth and compactly supported (in space), allowing us to take the limit of the estimate as $p \to \infty$. Applying Lemma 3.2.3 and Lemma 3.2.4 gives (3). $\square$

Appendix C

Notes On Chapter 4

Recall that in Step 1 of Section 4.4.1, $\Psi_{\epsilon_n} : \Omega \times [0,h] \to \mathbb{R}$ was extended to $\Psi_{\epsilon_n,E} : \mathbb{R}^3 \to \mathbb{R}$ and then mollified at length scale ϵ_n to produce a smooth velocity field $\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}$. We state and prove the following technical lemma regarding both the convergence of Ψ_{ϵ_n} and the mollified velocity fields $\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}$.

Lemma C.0.1. *1. Up to a subsequence, Ψ_{ϵ_n} converges strongly to Ψ in $C([0,T];\mathbb{H})$*

2. For any compact subdomain $\tilde{\Omega} \subset \Omega$, $\overline{\nabla}^{\perp}\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}$ converges strongly up to a subsequence to $\overline{\nabla}^{\perp}\Psi$ in $C([0,T];L^2(\tilde{\Omega} \times [0,h]))$.

Proof. To show (1), we consider Eq. (2.9) with $P_n = \Psi_{\epsilon_n} = Q_n$. By (4.28)

$$\sup_n \|\tilde{\nabla}\Psi_{\epsilon_n}\|_{L^\infty([0,T];H^{\frac{1}{2}}(\Omega\times[0,h]))} \leq C(\Omega,h,\lambda)\times \left(\|f_0\|_{L^2} + \|g_0\|_{L^2} + \|j_0\|_{L^2} + \|a_L\|_{L^1([0,T];L^2)} + \|a_\nu\|_{L^1([0,T];L^2)}\right).$$

Taking the trace then shows that

$$\sup_n \|\overline{\nabla}^{\perp}\Psi_{\epsilon_n}\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} \leq C(\Omega,h,\lambda)\times \left(\|f_0\|_{L^2} + \|g_0\|_{L^2} + \|j_0\|_{L^2} + \|a_L\|_{L^1([0,T];L^2)} + \|a_\nu\|_{L^1([0,T];L^2)}\right). \tag{C.1}$$

By construction of the extension $\Psi_{\epsilon_n,E}$,

$$\overline{\nabla}^{\perp}\Psi_{\epsilon_n,E}(x,y,z) = \begin{cases} \overline{\nabla}^{\perp}\Psi_{\epsilon_n}(z) & (x,y,z) \in \tilde{\Omega} \times [0,h] \\ \overline{\nabla}^{\perp}\Psi_{\epsilon_n}(0) & (x,y,z) \in \tilde{\Omega} \times (-\infty,0] \\ \overline{\nabla}^{\perp}\Psi_{\epsilon_n}(h) & (x,y,z) \in \tilde{\Omega} \times [h,\infty), \end{cases}$$

showing that $\overline{\nabla}^{\perp}\Psi_{\epsilon_n,E}$ is uniformly bounded in n in $L^\infty([0,T]\times[-\epsilon_n,h+\epsilon_n];L^2(\Omega))$. Therefore,

$$\sup_n \|\overline{\nabla}^{\perp}\Psi_{\epsilon_n,E}(t) * \eta_{\epsilon_n}\|_{L^\infty([0,T]\times[0,h];L^2(\Omega))} < \infty \tag{C.2}$$

Thus the assumptions of Eq. (2.9) are satisfied, and up to a subsequence, Ψ_{ϵ_n} converges to Ψ strongly in $C([0,T];\mathbb{H})$.

Moving to (2), let $\tilde{\Omega}$ be a fixed compact subdomain of Ω. We have that for $t \in [0,T]$,

$$\begin{aligned}
\limsup_{n\to\infty} \|\overline{\nabla}^{\perp}(\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}(t) - \Psi(t))\|_{L^2(\tilde{\Omega}\times[0,h])} &\leq \limsup_{n\to\infty} \|\overline{\nabla}^{\perp}(\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}(t) - \Psi_{\epsilon_n}(t))\|_{L^2(\tilde{\Omega}\times[0,h])} \\
&\quad + \limsup_{n\to\infty} \|\overline{\nabla}^{\perp}(\Psi_{\epsilon_n}(t) - \Psi(t))\|_{L^2(\tilde{\Omega}\times[0,h])} \\
\leq \sup_n \|\overline{\nabla}^{\perp}\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}(t)\|_{L^2(\tilde{\Omega}\times([0,\delta)\cup(h-\delta,h]))} &+ \sup_n \|\overline{\nabla}^{\perp}\Psi_{\epsilon_n}(t)\|_{L^2(\tilde{\Omega}\times([0,\delta)\cup(h-\delta,h]))} \\
+ \limsup_{n\to\infty} \|\overline{\nabla}^{\perp}(\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}(t) &- \Psi_{\epsilon_n}(t))\|_{L^2(\tilde{\Omega}\times[\delta,h-\delta])}.
\end{aligned}$$

By (C.1) and (C.2), the first two terms go to zero as $\delta \to 0$. So it suffices to show that for fixed δ that

$$\limsup_{n\to\infty} \|\overline{\nabla}^{\perp}(\Psi_{\epsilon_n,E} * \eta_{\epsilon_n}(t) - \Psi_{\epsilon_n}(t))\|_{L^2(\tilde{\Omega}\times[\delta,h-\delta])} = 0.$$

For n large enough,

$$\Psi_{\epsilon_n,E} * \eta_{\epsilon_n} = \Psi_{\epsilon_n} * \eta_{\epsilon_n} \qquad \forall (x,y,z) \in (\tilde{\Omega} \times [\delta, h-\delta]).$$

By extending Ψ_{ϵ_n} from $\tilde{\Omega}\times[\delta,h-\delta]$ to $\mathbb{R}^3$ using a standard Sobolev extension operator, it suffices to prove the claim for functions defined on all of $\mathbb{R}^3$. Using the Fourier characterization of $H^{\frac{3}{2}}(\mathbb{R}^3)$, we can write

$$\begin{aligned}
\|\overline{\nabla}^{\perp}\Psi_{\epsilon_n} * \eta_{\epsilon_n}(t) - \overline{\nabla}^{\perp}\Psi_{\epsilon_n}(t)\|^2_{L^2(\mathbb{R}^3)} &\leq \int_{\mathbb{R}^3} |\xi|^2 |\hat{\Psi}_{\epsilon_n}(t,\xi)|^2 |\hat{\eta}_{\epsilon_n}(\xi) - 1|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^3} |\xi|^2 (1+|\xi|^2)^{\frac{1}{2}} |\hat{\Psi}_{\epsilon_n}(t,\xi)|^2 \frac{|\hat{\eta}(\epsilon_n\xi) - 1|^2}{(1+|\xi|^2)^{\frac{1}{2}}} \, d\xi \\
&\leq \sup_n \|\Psi_{\epsilon_n}(t)\|^2_{H^{\frac{3}{2}}(\mathbb{R}^3)} \cdot \sup_\xi \frac{|\hat{\eta}(\epsilon_n\xi) - 1|^2}{(1+|\xi|^2)^{\frac{1}{2}}}.
\end{aligned}$$

which goes to zero uniformly in t as $n \to \infty$ since $\hat{\eta}$ is smooth and $\hat{\eta}(0) = 1$, concluding the proof. □

Appendix D

Notes On Chapter 5

For the proof of Theorem 5.2.1, we shall need several identities, definitions, and notations concerning Littlewood-Paley decompositions and Besov spaces. The homogeneous Besov spaces $\mathring{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)$ are defined via the usual bi-infinite sequence of homogeneous Littlewood-Paley decompositions (per the text of Bahouri, Chemin, and Danchin [3]). Here $\{\gamma_\epsilon\}_{\epsilon>0}$ is a sequence of compactly supported, radially symmetric approximate identities. For a function $u : \mathbb{R}^2 \to \mathbb{R}^n$, we define $u^\epsilon := u * \gamma_\epsilon$.

Proposition D.0.1. *1. For L^1_{loc} functions f and g,*

$$\int_{\mathbb{R}^2} g(f^\epsilon)^\epsilon = \int_{\mathbb{R}^2} g^\epsilon f^\epsilon$$

2. The following commutator identity holds:

$$(f \cdot g)^\epsilon(x) - f^\epsilon g^\epsilon(x) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(f(x-\bar{x}) - f(x)\right) \gamma_\epsilon(\bar{x}) \left(g(x-\bar{x}) - g(x-x')\right) \gamma_\epsilon(x')\, dx'\, d\bar{x}$$

3. For $\alpha \in (0,1)$ and $u \in \mathring{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)$, there exists C independent of u such that for all $|y| > 0$,

$$\|u(\cdot - y) - u(\cdot)\|_{L^3(\mathbb{R}^2)} \leq C y^\alpha \|u\|_{\mathring{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)}$$

and

$$\|\overline{\nabla} u^\epsilon\|_{L^3(\mathbb{R}^2)} \leq C \epsilon^{\alpha-1} \|u\|_{\mathring{B}^{\alpha}_{\infty,\infty}(\mathbb{R}^2)}.$$

Proof. (1) follows immediately from a change of variables and the radial symmetry of the mollifier. For (2), we can write

$$\begin{aligned}(f\cdot g)^\epsilon(x)-f^\epsilon g^\epsilon(x) &= \int_{\mathbb{R}^2} f(x-\bar{x})g(x-\bar{x})\gamma_\epsilon(\bar{x})\,d\bar{x}\\ &\quad -\int_{\mathbb{R}^2} f(x-\bar{x})\gamma_\epsilon(\bar{x})\,d\bar{x}\cdot\int_{\mathbb{R}^2} g(x-x')\gamma_\epsilon(x')\,dx'\\ &= \int_{\mathbb{R}^2} f(x-\bar{x})\gamma_\epsilon(\bar{x})\left(g(x-\bar{x})-\int_{\mathbb{R}^2} g(x-x')\gamma_\epsilon(x')\,dx'\right)d\bar{x}\\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(f(x-\bar{x})-f(x)\right)\gamma_\epsilon(\bar{x})\left(g(x-\bar{x})-g(x-x')\right)\gamma_\epsilon(x')\,dx'\,d\bar{x}\end{aligned}$$

Statements and proofs of (3) can be found in the text of Bahouri, Chemin, and Danchin [3]. □

Let us define the convolution and projection operators we shall make use of in Chapter 5. We divide them into two categories: kernels that depend on x, y, and z and therefore act on functions whose domain is $\mathbb{T}^3$, and kernels that depend only on x and y and therefore act on functions defined on $\mathbb{T}^2$. At various points throughout the discussion, we will freely substitute definitions and proofs for operators defined on $\mathbb{R}^n$ rather than $\mathbb{T}^n$. Standard transference arguments then provide analogous results for the periodic operators. In addition, all periodic functions are assumed to have mean zero. To simplify notation, we shall write sums over $\mathbb{Z}^3-\{0\}$ as simply being over $\mathbb{Z}^3$, and analogously for $\mathbb{Z}^2$. We begin with definitions and some facts about Hölder spaces.

Definition D.0.1 (Hölder Spaces). *Let $\alpha\in(0,1)$ and k a non-negative integer and $f:\mathbb{R}\times\mathbb{T}^n\to\mathbb{R}$ a function of time and space with mean value zero on $\mathbb{T}^n$ for each fixed time.*

1. *The integer spatial Hölder norms are defined by*

$$\|f\|_{C^k}=\sup_{t,x}\left|\nabla_x^k f(t,x)\right|.$$

2. *The non-integer spatial Hölder norms are defined by*

$$\|f\|_{C^{k,\alpha}}=\sup_{t,x,y}\frac{\left|\nabla_x^k f(t,x)-\nabla_x^k f(t,y)\right|}{|x-y|}+\|f\|_{C^k}.$$

3. *The following interpolation inequality holds for* $0 \leq r \leq 1$.

$$\|f\|_{C^{r\alpha}} \lesssim \|f\|_{C^0}^{1-r} \|f\|_{C^\alpha}^{r}.$$

We shall require a Bernstein inequality.

Lemma D.0.2 (**Bernstein Inequality**). *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a smooth function whose Fourier transform* $\hat{f}$ *vanishes in a neighborhood of the origin. If* $\hat{K}$ *is a Fourier multiplier which is smooth away from the origin and homogeneous of degree* s *and one of the following holds for* $\lambda > 0$,

1. $\operatorname{supp} \hat{f} \subset \{|\xi| \leq \lambda\}$ *and* $s > 0$
2. $\operatorname{supp} \hat{f} \subset \{|\xi| \geq \lambda\}$ *and* $s < 0$
3. $\operatorname{supp} \hat{f} \subset \{|\xi| \approx \lambda\}$ *and* $s \in \mathbb{R}$,

then

$$\left\| \left(\hat{K} \hat{f} \right)^{\vee} \right\|_{C^0} \lesssim \lambda^s \|f\|_{C^0}.$$

Definition D.0.2 ($\mathbb{T}^3$ **Operators**). *Let* $f : \mathbb{T}^3 \to \mathbb{R}$, $g : \mathbb{T}^3 \to \mathbb{R}^3$ *be smooth, mean-zero functions.*

1. *The vector of* $\mathbb{T}^3$*-Riesz transforms, denoted* $\mathcal{R}^3$, *acts on Fourier series via*

$$\mathcal{R}^3(f) = \sum_{\mathbb{Z}^3} \frac{ik}{|k|} \hat{f}(k) e^{ik\cdot x}$$

and satisfies

$$\left\|\mathcal{R}^3(f)\right\|_{C^\alpha} \lesssim \|f\|_{C^\alpha}$$

for non-integer $\alpha > 0$ *or for* f *with frequency support in an annulus. If* k *is an integer, then*

$$\left\|\mathcal{R}^3(f)\right\|_{C^k} \lesssim \|f\|_{C^{k+\alpha}}$$

where the implicit constant depends on $\alpha > 0$.

2. *The projector onto gradients $\mathbb{P}_\nabla$ is defined by*

$$\mathbb{P}_\nabla (g) := -\left(\mathcal{R}^3 \otimes \mathcal{R}^3\right)(g)$$

and satisfies the same estimates as $\mathcal{R}^3$.

3. *The projector onto curls $\mathbb{P}_{\mathrm{curl}}$ is defined by*

$$\mathbb{P}_{\mathrm{curl}}(g) = (\mathrm{Id} - \mathbb{P}_\nabla)(g) = \left(\mathrm{curl} \circ (-\Delta)^{-1} \circ \mathrm{curl}\right)(g)$$

and satisfies the same estimates as $\mathcal{R}^3$ and $\mathbb{P}_\nabla$.

4. *Let $\lambda \in \mathbb{N}$ and $k \in \mathbb{S}^2 \cap \mathbb{Q}$ such that $\lambda k \in \mathbb{Z}^3$. The projector $\mathbb{P}^\nabla_{\lambda,k}$ is defined by*

$$\mathbb{P}^\nabla_{\lambda,k}(g) = e^{i\lambda k\cdot x}\frac{ik}{|k|} \otimes \frac{ik}{|k|}\hat{g}(k)$$

and satisfies

$$\left\|\mathbb{P}^\nabla_{\lambda,k}(g)\right\|_{C^0} \lesssim \|g\|_{C^0}, \qquad \left\|\mathbb{P}^\nabla_{\lambda,k}(g)\right\|_{C^\alpha} \lesssim \|g\|_{C^0}\lambda^\alpha.$$

For $\lambda = \lambda_{q+1}$, we will denote this operator by $\mathbb{P}^\nabla_{q+1,k}$.

Definition D.0.3 (**$\mathbb{T}^2$ Operators**). *Let $f : \mathbb{T}^2 \to \mathbb{R}$, $g : \mathbb{T}^2 \to \mathbb{R}^2$ be smooth, mean-zero functions.*

1. *The vector of $\mathbb{T}^2$-Riesz transforms, denoted $\mathcal{R}^2$, acts on Fourier series via*

$$\mathcal{R}^2(f) = \sum_{\mathbb{Z}^2} \frac{ik}{|k|}\hat{f}(k)e^{ik\cdot x}$$

and satisfies

$$\left\|\mathcal{R}^2(f)\right\|_{C^\alpha} \lesssim \|f\|_{C^\alpha}$$

for non-integer $\alpha > 0$ or for f with frequency support in an annulus. If k is an integer, then

$$\left\|\mathcal{R}^2(f)\right\|_{C^k} \lesssim \|f\|_{C^{k+\alpha}}$$

where the implicit constant depends on $\alpha > 0$.

2. *The projector onto gradients* $\mathbb{P}^{\overline{\nabla}}$ *is defined by*

$$\mathbb{P}^{\overline{\nabla}}(g) = -\left(\mathcal{R}^2 \otimes \mathcal{R}^2\right)(g)$$

and satisfies the same estimates as $\mathcal{R}^2$*. When* $g = (g_1, g_2, g_3) : \mathbb{T}^3 \to \mathbb{R}^3$, $\mathbb{P}^{\overline{\nabla}}(g)$ *projects on the first two components and is the identity on the third component.*

3. *The projector onto perpendicular gradients* $\mathbb{P}^{\overline{\nabla}^\perp}$ *is defined by*

$$\mathbb{P}^{\overline{\nabla}^\perp}(g) = \left(\mathrm{Id} - \mathbb{P}^{\overline{\nabla}}\right)(g) = \left(\overline{\nabla}^\perp \circ \left(-\overline{\Delta}\right)^{-1} \circ \left(\overline{\nabla}^\perp \cdot\right)\right)(g)$$

and satisfies the same estimates as $\mathcal{R}^2$ *and* $\mathbb{P}^{\overline{\nabla}}$*. When* $g = (g_1, g_2, g_3) : \mathbb{T}^3 \to \mathbb{R}^3$, $\mathbb{P}^{\overline{\nabla}^\perp}(g)$ *projects on the first two components and is zero in the third component.*

4. *The inverse of* $\overline{\nabla}^\perp$*, denoted* $\left(\overline{\nabla}^\perp\right)^{-1}$*, is defined by*

$$\left(\overline{\nabla}^\perp\right)^{-1}(g) = \left(-\overline{\Delta}\right)^{-1} \circ \left(\overline{\nabla}^\perp \cdot\right)(g).$$

If the frequency support of g *is contained in an annulus of radius* λ*, then*

$$\left\| \left(\overline{\nabla}^\perp\right)^{-1} g \right\|_{C^0} \lesssim \frac{1}{\lambda} \|g\|_{C^0}.$$

5. *Let* $\lambda > 0$ *and define* $\overline{\mathbb{P}}_{\approx\lambda}$ *by*

$$\overline{\mathbb{P}}_{\approx\lambda}(f) = \sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} \hat{f}(k) e^{ik\cdot x}.$$

Define $\overline{\mathbb{P}}_{\leq\lambda}$ *by*

$$\overline{\mathbb{P}}_{\leq\lambda}(f) = \sum_{|k| \leq 2\lambda} \hat{f}(k) e^{ik\cdot x}$$

and $\overline{\mathbb{P}}_{\leq\lambda}$ *similarly. Each operator is bounded from* C^α *to* C^α *for any* $\alpha \geq 0$*.*

We shall frequently apply the $\mathbb{T}^2$ operators to functions $f : \mathbb{T}^3 \to \mathbb{R}$. If K is a $\mathbb{T}^2$ convolution operator, then by definition

$$K(f)(x, y, z) = \int_{\mathbb{T}^2} K(x - s, y - t)\, f(s, t, z)\, ds\, dt.$$

The following lemma will be applied repeatedly throughout the paper.

Lemma D.0.3. *Let $f:\mathbb{T}^3\to\mathbb{R}$ be a smooth function. Denote $k\in\mathbb{Z}^3$ by $\left(\bar{k},k_3\right)$ and let $\lambda>0$. Then $\overline{\mathbb{P}}_{\approx\lambda}(f)$ is supported in frequency in the cylinder*

$$\mathcal{C}_\lambda=\left\{k:|\bar{k}|\approx\lambda,k_3\in\mathbb{Z}\right\}.$$

If $\operatorname{supp}\hat{f}\subset\mathcal{C}_\lambda$*, then $\overline{\mathbb{P}}_{\approx\lambda}(f)=f$. Furthermore, analogous statements hold for $\overline{\mathbb{P}}_{\leq\lambda}$ and $\overline{\mathbb{P}}_{\geq\lambda}$ by replacing $\approx$ with $\leq$ and $\geq$, respectively.*

Proof. Fix $z\in[0,2\pi]$. For $(x,y,z)\in\mathbb{T}^3$, we denote $(x,y,0)$ by $\bar{x}$. Then

$$\overline{\mathbb{P}}_{\approx\lambda}(f)(x,y,z)=\sum_{\bar{k}\approx\lambda}\hat{f}(\bar{k},z)e^{i\bar{k}\cdot\bar{x}}$$

where

$$\hat{f}\left(\bar{k},z\right)=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}f(x,y,z)e^{i\bar{k}\cdot\bar{x}}\,dx\,dy.$$

Letting z vary, we have that $\hat{f}\left(\bar{k},z\right)$ is a smooth function of z and can therefore be written as

$$\hat{f}\left(\bar{k},z\right)=\sum_{k_3\in\mathbb{Z}}\hat{a}\left(\bar{k},k_3\right)e^{ik_3\cdot z}.$$

Combining the series, we have

$$\overline{\mathbb{P}}_{\approx\lambda}(f)(x,y,z)=\sum_{|\bar{k}|\approx\lambda}\sum_{k_3\in\mathbb{Z}}\hat{a}(\bar{k},k_3)e^{ik_3\cdot z}e^{i\bar{k}\cdot\bar{x}}=:\sum_{k\in\mathcal{C}_\lambda}\hat{a}(k)e^{ik\cdot x}.$$

By the uniqueness of $\mathbb{T}^3$ Fourier coefficients, if $\operatorname{supp}\hat{f}\subset\mathcal{C}_\lambda$, then $\hat{a}(k)=\hat{f}(k)$, and $\overline{\mathbb{P}}_{\approx\lambda}(f)=f$. $\square$

Here we collect several types of estimates which shall be necessary throughout the construction. All have become essentially standard in recent convex integration schemes. We begin with the following estimates for solutions to transport equations. For a proof, we refer the reader to [12].

Lemma D.0.4 (Transport Estimates)**.** *Consider the transport equation*

$$\partial_t f+u\cdot\nabla f=g,\qquad\qquad f|_{t_0}=f_0$$

where $f, g : \mathbb{T}^n \to \mathbb{R}$ and $u : \mathbb{T}^n \to \mathbb{R}^n$ are smooth functions. Let Φ be the inverse of the flow X of u defined by

$$\frac{d}{dt}X = u(X,t), \qquad X(x,t_0) = x.$$

Then the following hold:

1. $\|f(t)\|_{C^0} \leq \|f_0\|_{C^0} + \int_{t_0}^t \|g(s)\|_{C^0}\, ds$
2. $\|Df(t)\|_{C^0} \leq \|Df_0\|_{C^0} e^{(t-t_0)\|Du\|_{C^0}} + \int_{t_0}^t e^{(t-s)\|Du\|_{C^0}} \|Dg(s)\|_{C^0}\, ds$
3. *For any $N \geq 2$, there exists a constant $C = C(N)$ such that*

$$\begin{aligned}\|D^N f(t)\|_{C^0} \leq & \left(\|D^N f_0\|_{C^0} + C(t-t_0)\|D^n u\|_{C^0}\|Df\|_{C^0}\right) e^{C(t-t_0)\|Du\|_{C^0}} \\ & + \int_{t_0}^t e^{C(t-s)\|Du\|_{C^0}} \left(\|D^N g(s)\|_{C^0} + C(t-s)\|D^N u\|_{C^0}\|Dg(s)\|_{C^0}\right) ds\end{aligned}$$

4. $\|D\Phi(t) - \operatorname{Id}\|_{C^0} \leq e^{(t-t_0)\|Du\|_{C^0}} - 1 \leq (t-t_0)\|Du\|_{C^0} e^{(t-t_0)\|Du\|_{C^0}}$
5. *For $N \geq 2$ and a constant $C = C(N)$,*

$$\|D^N \Phi(t)\|_{C^0} \leq C(t-t_0)\|D^N u\|_{C^0} e^{C(t-t_0)\|Du\|_{C^0}}$$

The following estimate controls the norms of compositions of functions, particularly the perturbation.

Lemma D.0.5 (Chain Rule)**.** *Let $\Omega \subset \mathbb{R}^D$ $f : \Omega \to \mathbb{R}$, $g : \mathbb{R}^d \to \Omega$ be smooth functions. Then for every integer $N \geq 1$, there is a constant $C = C(N, d, D)$ such that*

$$\|D^N(f \circ g)\|_{C^0} \leq C\left(\|Df\|_{C^0}\|D^N g\|_{C^0} + \|Df\|_{C^{N-1}}\|g\|_{C^0}^{N-1}\|D^N g\|_{C^0}\right)$$

and

$$\|D^N(f \circ g)\|_{C^0} \leq C\left(\|Df\|_{C^0}\|D^N g\|_{C^0} + \|Df\|_{C^{N-1}}\|Dg\|_{C^0}^{N}\right).$$

We shall make use of the following commutator estimates. The estimate in Proposition D.0.6 is essentially contained in [30], although the version stated here is a slight alteration whose statement and proof can be found in [14]. The commutator estimate (D.1) for convolution operators localized in frequency can be found in [59] or [9]. The estimates (D.2) and (D.3) follow the methods of proof given in [59] and [9].

Proposition D.0.6. *Let $\alpha \in (0,1)$ and $N \geq 0$. Let T_K be a $\mathbb{R}^n$-Calderón-Zygmund operator with kernel K. Let $b \in C^{N+1,\alpha}(\mathbb{T}^n)$ be a vector field and $f \in C^{N,\alpha}(\mathbb{T}^n)$. Then there exists a constant $C = C(\alpha, N, K)$ such that*

$$\| [T_K, b \cdot \nabla] f \|_{N+\alpha} \leq C \|b\|_{1+\alpha} \|f\|_{N+\alpha} + \|b\|_{N+1+\alpha} \|f\|_{\alpha}.$$

Proposition D.0.7. *Let $s \in \mathbb{R}$, $\lambda \geq 1$, and let T_K be an order s convolution operator localized at length scale λ^{-1} whose action on smooth functions is given by convolution with a kernel K satisfying the bounds*

$$\| |x|^a \nabla^b K(x) \|_{L^1(\mathbb{R}^n)} \leq C(a,b) \lambda^{b-a+s}$$

for all $0 \leq a, |b|$. Then the following hold.

1. *For $f : \mathbb{T}^n \to \mathbb{C}$ a smooth function and $u : \mathbb{T}^n \to \mathbb{R}^n$ a smooth vector field with $\nabla \cdot u = 0$, we have*

$$\| [u \cdot \nabla, T_K] f \|_{C^0} \leq \lambda^s \|\nabla u\|_{C^0} \|f\|_{C^0} \tag{D.1}$$

2. *For $f : \mathbb{T}^n \to \mathbb{C}$ a smooth function and $u : \mathbb{T}^n \to \mathbb{R}^n$ a smooth vector field with $\nabla \cdot u = 0$, the iterated commutator $\left[\partial_t + u \cdot \overline{\nabla}, \left[u \cdot \overline{\nabla}, T_K\right]\right](f)$ obeys the estimate*

$$\left\| \left[\partial_t + u \cdot \overline{\nabla}, \left[u \cdot \overline{\nabla}, T_K\right]\right](f) \right\| \lesssim \lambda^{s-1} \|u\|_{C^1}^2 \|f\|_{C^1} + \|f\|_{C^0} \left(\lambda^{s+1} \|\partial_t u + u \cdot \overline{\nabla} u\|_{C^0} + \lambda^s \|u\|_{C^1}^2 \right). \tag{D.2}$$

3. *For $f, g : \mathbb{T}^n \to \mathbb{C}$ smooth functions, we have (for an implicit constant depending on k as well)*

$$\| [g, T_K] f \|_{C^k} \lesssim \lambda^{s-1} \sum_{0 \leq j \leq k} \|\nabla g\|_{C^j} \|f\|_{C^{k-j}}. \tag{D.3}$$

Proof. The proof of (1) is contained in the appendix of [9]. Moving on to the iterated commutator estimate of (2), we first write

$$\begin{aligned} \left[u \cdot \overline{\nabla}, T_K\right](f) &= u(x) \cdot \overline{\nabla} \int_{\mathbb{R}^3} K(y) f(x-y)\, dy - \int_{\mathbb{R}^3} K(y) u(x-y) \cdot \overline{\nabla} f(x-y)\, dy \\ &= \int_{\mathbb{R}^3} f(x-y) \overline{\nabla} K(y) \cdot (u(x) - u(x-y)). \end{aligned}$$

Now expanding the iterated commutator, we have

$$
\begin{aligned}
&\left[\partial_t + u\cdot\overline{\nabla}, \left[u\cdot\overline{\nabla}, T_K\right]\right](f) = \\
&\quad \left(\partial_t + u(x)\cdot\overline{\nabla}\right)\left(\int_{\mathbb{R}^3} f(x-y)\overline{\nabla}K(y)\cdot(u(x)-u(x-y))\right)dy \\
&\qquad - \int_{\mathbb{R}^3}\left(\partial_t f(x-y) + u(x-y)\cdot\overline{\nabla}f(x-y)\right)\overline{\nabla}K(y)\cdot(u(x)-u(x-y))\,dy \\
&\quad = \int_{\mathbb{R}^3}\left((u(x)-u(x-y))\cdot\overline{\nabla}f(x-y)\right)\overline{\nabla}K(y)\cdot(u(x)-u(x-y))\,dy \\
&\qquad + \int_{\mathbb{R}^3} f(x-y)\overline{\nabla}K(y)\cdot\left(\partial_t u(x) + u(x)\cdot\overline{\nabla}u(x) - \partial_t u(x-y) - u(x)\cdot\overline{\nabla}u(x-y)\right)dy \\
&\quad =: I + II.
\end{aligned}
$$

Estimating I first, we write

$$
\begin{aligned}
I &\leq \int_{\mathbb{R}^3}|\overline{\nabla}K(y)|\|u\|_{C^1}^2|y|^2\|f\|_{C^1}\,dy \\
&\leq \|u\|_{C^1}^2\|f\|_{C^1}\lambda^{s-1}.
\end{aligned}
$$

Before estimating II, note that

$$
\partial_t u(x-y) + u(x)\cdot\overline{\nabla}u(x-y) = \left(\left(\partial_t + u\cdot\overline{\nabla}\right)u\right)(x-y) + \left((u(x)-u(x-y))\cdot\overline{\nabla}u(x-y)\right).
$$

Therefore,

$$
\begin{aligned}
II2 &\leq \int_{\mathbb{R}^3}\|f\|_{C^0}|\overline{\nabla}K(y)|\left(\|\partial_t u + u\cdot\overline{\nabla}u\|_{C^0} + |y|\|u\|_{C^1}^2\right)dy \\
&\lesssim \|f\|_{C^0}\left(\lambda^{s+1}\|\partial_t u + u\cdot\overline{\nabla}u\|_{C^0} + \lambda^s\|u\|_{C^1}^2\right).
\end{aligned}
$$

Combining the estimates gives the result.

To prove (3), we follow the idea from [9] and write that

$$
\begin{aligned}
\left|\nabla^k\left(T_K(bf)(x) - b(x)T_K f(x)\right)\right| &= \left|\int_{\mathbb{R}^n}\nabla^k\left((b(x)-b(x-y))f(x-y)\right)K(y)\,dy\right| \\
&= \left|\int_{\mathbb{R}^n}\nabla^k\left(\left(\int_0^1\nabla b(x-sy)\,ds\right)\cdot yf(x-y)\right)K(y)\,dy\right|.
\end{aligned}
$$

Applying the Leibniz rule and using the integrability assumption on K finishes the proof. □

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